Discrete nonholonomic mechanics on Lie groups

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Outline

Introduction: Continuous Nonholonomic Mechanical Systems

Discrete Nonholonomic Mechanics

LL systems

Suslov Problem
Outline

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Discrete Nonholonomic Mechanics

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Suslov Problem
Nonholonomic Mechanical System \((Q, \mathcal{L}, \mathcal{D})\)

- Configuration space \(Q\), a smooth \(n\) dimensional manifold.
Nonholonomic Mechanical System \((Q, \mathcal{L}, \mathcal{D})\)

- Configuration space \(Q\), a smooth \(n\) dimensional manifold.
- Lagrangian \(\mathcal{L} : TQ \to \mathbb{R}\)
  
  \(\mathcal{L} = \text{Kinetic energy} - \text{Potential energy}\)
Nonholonomic Mechanical System \((Q, \mathcal{L}, \mathcal{D})\)

- Configuration space \(Q\), a smooth \(n\) dimensional manifold.

- Lagrangian \(\mathcal{L} : TQ \to \mathbb{R}\)
  \(\mathcal{L} = \text{Kinetic energy} - \text{Potential energy}\)

- A non-integrable constraint distribution \(\mathcal{D} \subset TQ\) defined by \(m < n\) constraints that are linear and homogeneous in the velocities:

  \[\sum_{s=1}^{n} \beta^j_s(q) \dot{q}^s = 0, \quad j = 1, \ldots, m.\]
Nonholonomic Mechanical System \((Q, \mathcal{L}, D)\)

- Configuration space \(Q\), a smooth \(n\) dimensional manifold.
- Lagrangian \(\mathcal{L} : TQ \to \mathbb{R}\)
  \(\mathcal{L} = \text{Kinetic energy - Potential energy}\)
- A non-integrable constraint distribution \(D \subset TQ\) defined by \(m < n\) constraints that are linear and homogeneous in the velocities:

\[
\sum_{s=1}^{n} \beta_{s}^{j}(q) \dot{q}^{s} = 0, \quad j = 1, \ldots, m.
\]

\(D_{q} \subset T_{q}Q\) is the annihilator of the one-forms on \(Q\):

\[
\beta^{j} = \sum_{s=1}^{n} \beta_{s}^{i}(q) dq^{s}, \quad j = 1, \ldots, m.
\]
Equations of motion (Lagrangian formulation)

Lagrange-D’Alembert principle (Newton's Law):

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda_1 \beta_1(q) + \cdots + \lambda_m \beta_m(q).
\]

Reaction Forces
Equations of motion (Lagrangian formulation)

Lagrange-D’Alembert principle (Newton’s Law):

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\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda_1 \beta^1(q) + \cdots + \lambda_m \beta^m(q).
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Suslov Problem (1902) \( \Omega \cdot a = 0. \)
Equations of motion (Lagrangian formulation)

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Reaction Forces

\[
\dot{M} = M \times \Omega
\]
Equations of motion (Lagrangian formulation)

Lagrange-D’Alembert principle (Newton’s Law):

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\]

\[\text{Reaction Forces}\]

\[\dot{M} = M \times \Omega + \lambda a\]
Equations of motion (Lagrangian formulation)

Lagrange-D’Alembert principle (Newton’s Law):

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda_1 \beta_1(q) + \cdots + \lambda_m \beta_m(q). \]

Reaction Forces

\[ \dot{M} = M \times \Omega + \lambda a \quad \Omega \cdot a = 0 \]
Equations of motion (Lagrangian formulation)

Lagrange-D’Alembert principle (Newton’s Law):
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\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda_1 \beta^1(q) + \cdots + \lambda_m \beta^m(q) .
\]

Reaction Forces

Lagrange Multipliers \( \lambda_i \) determined uniquely by the constraints.

\[
\beta^j(q) \cdot \dot{q} = 0, \quad j = 1, \ldots, m \iff \dot{q} \in \mathcal{D}_q .
\]
Equations of motion (Lagrangian formulation)

Lagrange-D’Alembert principle (Newton’s Law):

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\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda_1 \beta^1(q) + \cdots + \lambda_m \beta^m(q).
\]

\( \lambda_i \) determined uniquely by the constraints.

\[ \beta^j(q) \cdot \dot{q} = 0, \quad j = 1, \ldots, m \iff \dot{q} \in D_q. \]

Conservation of Energy.
Equations of motion (Lagrangian formulation)

Lagrange-D’Alembert principle (Newton’s Law):

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda_1 \beta^1(q) + \cdots + \lambda_m \beta^m(q). \]

Reaction Forces

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\[ \beta^j(q) \cdot \dot{q} = 0, \quad j = 1, \ldots, m \iff \dot{q} \in D_q. \]

Conservation of Energy.

Equations are \textit{not} variational in the usual sense.
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Discrete Lagrange-D’Alembert principle

Lagrange-D’Alembert principle:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta^j(q),$$

$$\dot{q} \in D_q.$$ 

Continuous data \((Q, D, L)\).
Discrete Lagrange-D’Alembert principle

Lagrange-D’Alembert principle:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta^j(q),
\]

\[\dot{q} \in \mathcal{D}_q.\]

Continuous data \((Q, \mathcal{D}, L)\).

J. Cortés, S. Martínez (2001)

\(TQ \rightarrow Q \times Q\).
Discrete Lagrange-D’Alembert principle

Lagrange-D’Alembert principle:

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta_j^j(q), \]

\( \dot{q} \in \mathcal{D}_q. \)

Continuous data \((Q, \mathcal{D}, \mathcal{L}).\)

J. Cortés, S. Martínez (2001)

\( TQ \rightarrow Q \times Q. \)

\( \mathcal{L} \rightarrow \text{Discrete Lagrangian } \mathcal{L}_d : Q \times Q \rightarrow \mathbb{R}. \)
Discrete Lagrange-D’Alembert principle

Lagrange-D’Alembert principle:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta_j(q), \\
\dot{q} \in \mathcal{D}_q.
\]

Continuous data \((Q, \mathcal{D}, \mathcal{L})\).

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\(TQ \rightarrow Q \times Q\).

\(\mathcal{L} \rightarrow \text{Discrete Lagrangian } \mathcal{L}_d : Q \times Q \rightarrow \mathbb{R}\).

\(\mathcal{D} \subset TQ \rightarrow \text{Discrete constraint space } \mathcal{D}_d \subset Q \times Q\).
Discrete Lagrange-D’Alembert principle

Lagrange-D’Alembert principle:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta^j(q),
\]

\[\dot{q} \in \mathcal{D}_q.\]

Continuous data \((Q, \mathcal{D}, \mathcal{L})\).

J. Cortés, S. Martínez (2001)

\[TQ \longrightarrow Q \times Q.\]

\[\mathcal{L} \longrightarrow \text{Discrete Lagrangian } \mathcal{L}_d : Q \times Q \rightarrow \mathbb{R}.\]

\[\mathcal{D} \subset TQ \longrightarrow \text{Discrete constraint space } \mathcal{D}_d \subset Q \times Q.\]

\[(q, q) \in \mathcal{D}_d \text{ for all } q \in Q.\]
Discrete Lagrange-D’Alembert principle

Lagrange-D’Alembert principle:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta_j(q),
\]

\[\dot{q} \in D_q.\]

Continuous data \((Q, D, L)\).

J. Cortés, S. Martínez (2001)

\[TQ \rightarrow Q \times Q.\]

\[L \rightarrow \text{Discrete Lagrangian } L_d : Q \times Q \rightarrow \mathbb{R}.\]

\[D \subset TQ \rightarrow \text{Discrete constraint space } D_d \subset Q \times Q.\]

\[(q, q) \in D_d \text{ for all } q \in Q.\]

Discrete data \((Q, D_d, L_d, D)\).
Discrete Lagrange-D’Alembert principle

Lagrange-D’Alembert principle:

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\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta^j(q),
\]

\(\dot{q} \in D_q, \quad (\delta q(t) \in D_{q(t)})\)

Continuous data \((Q, D, L)\).
Discrete Lagrange-D’Alembert principle

Lagrange-D’Alembert principle:

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\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta^j(q),
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Continuous data \((Q, \mathcal{D}, \mathcal{L})\). Discrete data \((Q, \mathcal{D}_d, \mathcal{L}_d, \mathcal{D})\)
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\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = \sum_{j=1}^{m} \lambda_j \beta_j(q),
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Continuous data \((Q, \mathcal{D}, \mathcal{L})\). Discrete data \((Q, \mathcal{D}_d, \mathcal{L}_d, \mathcal{D})\)

Discrete Lagrange-D’Alembert principle:

\[
D_1 \mathcal{L}_d(q_k, q_{k+1}) + D_2 \mathcal{L}_d(q_{k-1}, q_k) = \sum_{j=1}^{m} \lambda_j^{(k)} \beta_j(q_k),
\]

\((q_k, q_{k+1}) \in \mathcal{D}_d \quad (\delta q_k \in \mathcal{D}_{q_k})\)
Discrete Lagrange-D’Alembert principle

Discrete data \((Q, D_d, L_d, D)\)

Discrete Lagrange-D’Alembert principle:

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D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = \sum_{j=1}^{m} \lambda_j^{(k)} \beta^j(q_k),
\]

\((q_k, q_{k+1}) \in D_d, \quad (\delta q_k \in D_{q_k}).\)

Notice that it is not a variational principle.

\[
S_d = \sum_{k=k_0}^{k_1-1} L_d(q_k, q_{k+1}), \quad q_{k_0}, q_{k_1}, \text{fixed.}
\]
Discrete Lagrange-D’Alembert principle

Discrete data \((Q, \mathcal{D}_d, \mathcal{L}_d, \mathcal{D})\)

Discrete Lagrange-D’Alembert principle:

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S_d = \sum_{k=k_0}^{k_1-1} \mathcal{L}_d(q_k, q_{k+1}), \quad q_{k_0}, q_{k_1}, \text{fixed.}
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Preserves momentum for horizontal symmetries.
Discrete Lagrange-D’Alembert principle

Discrete data \((Q, D_d, L_d, D)\)

Discrete Lagrange-D’Alembert principle:

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\]

Preserves momentum for \textit{horizontal symmetries}.

Does not preserve energy in general.
How to select discrete data?

Given \((D, L)\) select \((D_d, L_d)\).
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Configuration space is a Lie group $Q = G$. 

LL Invariance: Lagrangian and constraints can be written solely in terms of $\Omega$.

Kinetic energy metric is defined by an inertia tensor $I$:

$$L(g, \dot{g}) = \frac{1}{2} \int I \Omega, \Omega \, dt$$

Constraint distribution is defined by a subspace $d \subset g$ that is not a sub-algebra.
LL systems

Configuration space is a Lie group $Q = G$.

Left action
LL systems

Configuration space is a Lie group $Q = G$. Left action

$$G \times TG \rightarrow TG$$

$$h \cdot (g, \dot{g}) \mapsto (hg, h\dot{g})$$
LL systems

Configuration space is a Lie group \( Q = G \).

Left action

\[
G \times TG \to TG
\]

\[
h \cdot (g, \dot{g}) \mapsto (hg, h\dot{g})
\]

\[
TG/G \cong g, \quad (g, \dot{g}) \mapsto g^{-1}\dot{g} = \Omega.
\]
LL systems

Configuration space is a Lie group $Q = G$.
Left action

$$G \times TG \rightarrow TG$$

$$h \cdot (g, \dot{g}) \mapsto (hg, h\dot{g})$$

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LL Invariance: Lagrangian and constraints can be written solely in terms of $\Omega$. 
LL systems

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LL Invariance: Lagrangian and constraints can be written solely in terms of $\Omega$.
Kinetic energy metric is defined by an inertia tensor $\Pi : \mathfrak{g} \rightarrow \mathfrak{g}^*$. 
LL systems

Configuration space is a Lie group $Q = G$. Left action

$$G \times TG \rightarrow TG$$

$$h \cdot (g, \dot{g}) \mapsto (hg, h\dot{g})$$

$$TG/G \cong \mathfrak{g}, \quad (g, \dot{g}) \mapsto g^{-1}\dot{g} = \Omega.$$  

LL Invariance: Lagrangian and constraints can be written solely in terms of $\Omega$.

Kinetic energy metric is defined by an inertia tensor $\Pi: \mathfrak{g} \rightarrow \mathfrak{g}^*$. $\mathcal{L}(g, \dot{g}) = \frac{1}{2} \langle \Pi \Omega, \Omega \rangle = \ell(\Omega)$. 
LL systems

Configuration space is a Lie group \( Q = G \).

Left action

\[
G \times TG \to TG
\]

\[
h \cdot (g, \dot{g}) \mapsto (hg, h\dot{g})
\]

\[
TG/G \cong \mathfrak{g}, \quad (g, \dot{g}) \mapsto g^{-1}\dot{g} = \Omega.
\]

LL Invariance: Lagrangian and constraints can be written solely in terms of \( \Omega \).

Kinetic energy metric is defined by an inertia tensor

\[
\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*.
\]

\[
\mathcal{L}(g, \dot{g}) = \frac{1}{2} \langle \mathbb{I}\Omega, \Omega \rangle = \ell(\Omega).
\]

Constraint distribution is defined by a subspace \( \mathfrak{d} \subset \mathfrak{g} \) that is not a sub-algebra.

\[
\mathcal{D}_g = (L_g)^* \mathfrak{d}
\]
LL systems

Configuration space is a Lie group $Q = G$. Left action

\[ G \times TG \to TG \]
\[ h \cdot (g, \dot{g}) \mapsto (hg, h\dot{g}) \]

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LL Invariance: Lagrangian and constraints can be written solely in terms of $\Omega$.

Kinetic energy metric is defined by an inertia tensor $\mathbb{I} : g \to g^*$. $\mathcal{L}(g, \dot{g}) = \frac{1}{2} \langle \mathbb{I} \Omega, \Omega \rangle = \ell(\Omega)$.

Constraint distribution is defined by a subspace $\mathfrak{d} \subset g$ that is not a sub-algebra.

\[ \mathcal{D}_g = (L_g)^* \mathfrak{d} \]

$\mathfrak{d}$ is the annihilator of $a^1, \ldots, a^m \in g^*$. 
LL systems

Configuration space is a Lie group \( Q = G \).
Left action

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G \times TG \rightarrow TG
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h \cdot (g, \dot{g}) \mapsto (hg, h\dot{g})
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TG/G \cong \mathfrak{g}, \quad (g, \dot{g}) \mapsto g^{-1}\dot{g} = \Omega.
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LL Invariance: Lagrangian and constraints can be written solely in terms of \( \Omega \).
Kinetic energy metric is defined by an inertia tensor
\( \mathbb{I} : g \rightarrow g^* \). \( \mathcal{L}(g, \dot{g}) = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega \rangle = \ell(\Omega) \).
Constraint distribution is defined by a subspace \( \mathfrak{d} \subset g \) that is not a sub-algebra.

\[
\mathcal{D}_g = (L_g)^*\mathfrak{d}
\]

\( \mathfrak{d} \) is the annihilator of \( a^1, \ldots, a^m \in g^* \). \( \langle a^j, \Omega \rangle = 0 \).
Euler-Poincaré-Suslov equations

\[ \dot{M} = \text{ad}^*_\Omega M + \sum_{j=1}^{m} \lambda_j a^j \]

\[ \Omega = g^{-1} \dot{g}, \quad M = \mathbb{I} \Omega \]
Euler-Poincaré-Suslov equations

\[ \dot{M} = \text{ad}^*_\Omega M + \sum_{j=1}^m \lambda_j a_j \]

\[ \Omega = g^{-1} \dot{g}, \quad M = \mathcal{I} \Omega \]

Constraints:

\[ \langle a^j, \Omega \rangle = 0 \quad \Omega \in \mathfrak{d} \]

\[ \langle M, \mathcal{I}^{-1} a^j \rangle = 0, \quad M \in \mathcal{I}(\mathfrak{d}) = \mathfrak{u} \subset \mathfrak{g}^* \]
Euler-Poincaré-Suslov equations

\[ \dot{M} = \text{ad}_{\Omega}^* M + \sum_{j=1}^{m} \lambda_j a^j \]

\[ \Omega = g^{-1} \dot{g}, \quad M = \mathbb{I} \Omega \]

Constraints:

\[ \langle a^j, \Omega \rangle = 0 \quad \Omega \in \mathfrak{d} \]

\[ \langle M, \mathbb{I}^{-1} a^j \rangle = 0, \quad M \in \mathbb{I}(\mathfrak{d}) = u \subset \mathfrak{g}^* \]

**Key Observation:** Euler-Poincaré-Suslov equations are defined in a linear subspace of $\mathfrak{g}^*$. 
Discrete LL systems

Fedorov, Zenkov, 2005

Discrete Left action

\[ G \times (G \times G) \rightarrow G \times G \]

\[ h \cdot (g_k, g_{k+1}) \mapsto (hg_k, hg_{k+1}) \]

Discrete LL Invariance: Discrete Lagrangian and constraints can be written solely in terms of \( W_k \).

\[ L_d(g_k, g_{k+1}) = \int d(W_k) \]

\( Dd = \{ (g_k, g_{k+1}) \in G \times G : W_k \in S \subset G \} \)

Fedorov-Zenkov: \( S = \exp(d) \)

Our investigation: \( S = \text{Cayley}(d) \)
Discrete LL systems

Fedorov, Zenkov, 2005

Discrete Left action

\[ G \times (G \times G) \to G \times G \]
\[ h \cdot (g_k, g_{k+1}) \mapsto (hg_k, hg_{k+1}) \]

\[ (G \times G)/G \cong G, \quad (g_k, g_{k+1}) \mapsto g_k^{-1}g_{k+1} = W_k. \]
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Discrete LL Invariance: Discrete Lagrangian and constraints can be written solely in terms of \( W_k \).

\[ \mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k), \quad \ell_d : G \rightarrow \mathbb{R} \]
Discrete LL systems

Fedorov, Zenkov, 2005

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\[ G \times (G \times G) \to G \times G \]

\[ h \cdot (g_k, g_{k+1}) \mapsto (hg_k, hg_{k+1}) \]

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\[ \mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k), \quad \ell_d : G \to \mathbb{R} \]

\[ \mathcal{D}_d = \{(g_k, g_{k+1}) \in G \times G : W_k \in S \subset G\} \]
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Discrete LL Invariance: Discrete Lagrangian and constraints can be written solely in terms of \( W_k \).

\[ \mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k), \quad \ell_d : G \rightarrow \mathbb{R} \]

\[ \mathcal{D}_d = \{(g_k, g_{k+1}) \in G \times G : W_k \in S \subset G\} \]

Fedorov-Zenkov: \( S = \exp(\partial) \)
Discrete LL systems

Fedorov, Zenkov, 2005

Discrete Left action

\[ G \times (G \times G) \rightarrow G \times G \]

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Discrete LL Invariance: Discrete Lagrangian and constraints can be written solely in terms of \( W_k \).

\[ \mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k), \quad \ell_d : G \rightarrow \mathbb{R} \]

\[ \mathcal{D}_d = \{(g_k, g_{k+1}) \in G \times G : W_k \in S \subset G\} \]

Fedorov-Zenkov: \( S = \exp(\vartheta) \)

Our investigation: \( S = \text{Cayley}(\vartheta) \)
Discrete Euler-Poincaré-Suslov equations

Discrete Lagrange-D’Alembert Principle

\[ D_1 \mathcal{L}_d(g_k, g_{k+1}) + D_2 \mathcal{L}_d(g_{k-1}, g_k) = \sum_{j=1}^{m} \lambda_j^{(k)} L^*_{g_{k-1}} a^j \]
Discrete Euler-Poincaré-Suslov equations

Discrete Lagrange-D’Alembert Principle

\[ D_1 L_d(g_k, g_{k+1}) + D_2 L_d(g_{k-1}, g_k) = \sum_{j=1}^{m} \lambda_j^{(k)} L_{g_k}^* a^j \]

Pull-it back to \( g^* \) by left translation (Bobenko-Suris 1999).

\[ - \text{Ad}_{W_k^{-1}} M_{k+1} + M_k = \sum_{j=1}^{m} \lambda_j^{(k)} a^j \]

\[ M_k = L_{W_{k-1}}^* l'_d(W_{k-1}) \in g^* \]
Discrete Euler-Poincaré-Suslov equations

\[-\text{Ad}^*_{W_k^{-1}} M_{k+1} + M_k = \sum_{j=1}^{m} \lambda_j^{(k)} a^j\]

\[M_k = L^*_{W_{k-1}} \ell'_d(W_{k-1}) \in \mathfrak{g}^*\]
Discrete Euler-Poincaré-Suslov equations

\[- \text{Ad}_{W_k} M_{k+1} + M_k = \sum_{j=1}^{m} \lambda^{(k)}_j a^j\]

\[M_k = L^*_L \ell_d(W_{k-1}) \in g^*\]

Constraints:

\[W_k \in S\]

\[M_{k+1} = L^*_W \ell_d(W_k) \in U \subset g^*\]
Discrete Euler-Poincaré-Suslov equations

\[- \text{Ad}_{W_k^{-1}} M_{k+1} + M_k = \sum_{j=1}^{m} \lambda_j^{(k)} a^j\]

\[M_k = L^*_{W_{k-1}} \ell'_d(W_{k-1}) \in g^*\]

Constraints:

\[W_k \in S\]

\[M_{k+1} = L^*_{W_k} \ell'_d(W_k) \in \mathcal{U} \subset g^*\]

- **Key Observation:** Discrete Euler-Poincaré-Suslov equations are defined in the *Momentum Locus* \(\mathcal{U} \subset g^*\) - a nonlinear subset of \(g^*\).
Outline

Introduction: Continuous Nonholonomic Mechanical Systems

Discrete Nonholonomic Mechanics

LL systems

Suslov Problem
Suslov Problem

$$Q = G = SO(3)$$

Nonholonomic constraint:

$$\Omega \cdot a = 0,$$

Lagrangian:

$$\mathcal{L}(g, \dot{g}) = \ell(\Omega) = \frac{1}{2} \Omega \cdot \Omega$$
Suslov Problem

\[ Q = G = SO(3) \]

Select body frame:

\[ \mathbf{a} = e_3, \quad \mathbf{I} = \begin{pmatrix} l_{11} & 0 & l_{13} \\ 0 & l_{22} & l_{23} \\ l_{13} & l_{23} & l_{33} \end{pmatrix} \]

\[ \Omega_3 = 0, \quad \ell(\mathbf{\Omega}) = \frac{1}{2} \mathbf{I} \mathbf{\Omega} \cdot \mathbf{\Omega} \]
Suslov Problem - Reduced Dynamics

Reduced equations:

\[ \frac{d}{dt} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} -\Omega_2 (l_{13} \Omega_1 + l_{23} \Omega_2) \\ \frac{l_{11}}{\Omega_1 (l_{13} \Omega_1 + l_{23} \Omega_2)} \\ \frac{l_{12}}{l_{22}} \end{pmatrix} \]

Key aspects: Integrable, Hamiltonian, asymptotic behavior, no global preserved measure.
Suslov Problem - Reduced Dynamics

Reduced equations:

\[ \frac{d}{dt} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} -\Omega_2(l_{13}\Omega_1 + l_{23}\Omega_2) \\ \frac{l_{11}}{\Omega_1(l_{13}\Omega_1 + l_{23}\Omega_2)} \\ \frac{l_{22}}{\Omega_1(l_{13}\Omega_1 + l_{23}\Omega_2)} \end{pmatrix} \]

Key aspects: Integrable, Hamiltonian, asymptotic behavior, no global preserved measure.

Discrete Suslov Problem

Fedorov, Zenkov, 2005

Discrete constraint space

\[ W_k \in \mathcal{S} = \exp(\mathfrak{a}) \]

\[ \mathfrak{a} = \{ \Omega \in \mathbb{R}^3 : \Omega_3 = 0 \} \]
Discrete Suslov Problem

Fedorov, Zenkov, 2005

Discrete constraint space

\[ W_k \in S = \exp(\vartheta) \]

\[ \vartheta = \{ \Omega \in \mathbb{R}^3 : \Omega_3 = 0 \} \]

\( S \) = Matrices in \( SO(3) \) with axis of rotation

\[
\begin{pmatrix}
\Omega_1 \\
\Omega_2 \\
0
\end{pmatrix}
\]

\( S \cong \mathbb{RP}^2 \subset SO(3) \).
Discrete Suslov Problem

Our approach

Discrete constraint space

\[ W_k \in S = \text{Cayley}(\mathfrak{d}) \]
\[ \mathfrak{d} = \{ \Omega \in \mathbb{R}^3 : \Omega_3 = 0 \} \]
Discrete Suslov Problem

Our approach

Discrete constraint space

\[ W_k \in S = \text{Cayley}(\mathfrak{d}) \]
\[ \mathfrak{d} = \{ \Omega \in \mathbb{R}^3 : \Omega_3 = 0 \} \]

Cayley Transformation

\[
\text{Cayley} : \mathbb{R}^3 \cong \mathfrak{so}(3) \to SO(3)
\]
\[ \Omega \cong \hat{\Omega} \mapsto \left( E - \frac{1}{2} \hat{\Omega} \right) \left( E + \frac{1}{2} \hat{\Omega} \right)^{-1} \]
Discrete Suslov Problem

Our approach

Discrete constraint space

\[ \mathcal{W}_k \in \mathcal{S} = \text{Cayley}(\mathfrak{d}) \]
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\[ \text{Cayley} : \mathbb{R}^3 \cong \mathfrak{so}(3) \rightarrow SO(3) \]
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\[ \text{Cayley}(\Omega) = \text{Rotation with axis } \Omega \text{ and angle } \theta \]
\[ \cos \theta = \frac{4 - ||\Omega||^2}{4 + ||\Omega||^2}, \quad 0 \leq \theta < \pi. \]
Discrete Suslov Problem

Our approach

Discrete constraint space

\[ \mathcal{W}_k \in S = \text{Cayley}(\mathfrak{d}) \]
\[ \mathfrak{d} = \{ \Omega \in \mathbb{R}^3 : \Omega_3 = 0 \} \]

Cayley Transformation

\[ \text{Cayley} : \mathbb{R}^3 \cong \mathfrak{so}(3) \rightarrow SO(3) \]
\[ \Omega \cong \hat{\Omega} \mapsto \left( E + \frac{1}{2} \hat{\Omega} \right) \left( E - \frac{1}{2} \hat{\Omega} \right)^{-1} \]

Inverse Cayley Transformation

\[ \text{Cayley}^{-1} : SO(3) \setminus \mathcal{R}_\pi \rightarrow \mathfrak{so}(3) \cong \mathbb{R}^3 \]
\[ \mathcal{W} \mapsto 2(\mathcal{W} - E)(\mathcal{W} + E)^{-1}. \]
Discrete Suslov Problem

Our approach

Discrete constraint space

\[ W_k \in S = \text{Cayley}(\vartheta) \]
\[ \vartheta = \{ \Omega \in \mathbb{R}^3 : \Omega_3 = 0 \} \]

\[ S = \text{Matrices in } SO(3) \text{ with axis of rotation } \left( \begin{array}{c} \Omega_1 \\ \Omega_2 \\ 0 \end{array} \right) \text{ and angle of rotation } 0 \leq \theta < \pi. \]
Discrete Suslov Problem

Our approach

Discrete constraint space

\[ W_k \in \mathcal{S} = \text{Cayley}(\vartheta) \]
\[ \vartheta = \{ \Omega \in \mathbb{R}^3 : \Omega_3 = 0 \} \]

\[ \mathcal{S} = \text{Matrices in } SO(3) \text{ with axis of rotation } \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ 0 \end{pmatrix} \text{ and angle of rotation } 0 \leq \theta < \pi. \]

\[ \mathbb{R}^2 \cong \vartheta \cong \mathcal{S} \subset SO(3). \]
Choice of Discrete Lagrangian

$$\ell(\Omega) = \frac{1}{2} \langle \mathbb{I} \Omega, \Omega \rangle = \frac{1}{2} \text{Trace} \left( \Omega(\hat{\Omega})^T \mathbb{J} \right) = \ell(\hat{\Omega})$$

$$\mathbb{J} = \frac{1}{2} \begin{pmatrix}
  l_{22} + l_{33} - l_{11} & 0 & -l_{13} \\
  0 & l_{11} + l_{33} - l_{22} & -l_{23} \\
  -l_{13} & -l_{23} & l_{22} + l_{11} - l_{33}
\end{pmatrix}$$
Choice of Discrete Lagrangian

\[
\ell(\Omega) = \frac{1}{2} \langle \mathbb{I} \Omega, \Omega \rangle = \frac{1}{2} \text{Trace} \left( \hat{\Omega} (\hat{\Omega})^T \mathbb{J} \right) = \ell(\hat{\Omega})
\]

\[
\mathbb{J} = \frac{1}{2} \begin{pmatrix}
    l_{22} + l_{33} - l_{11} & 0 & -l_{13} \\
    0 & l_{11} + l_{33} - l_{22} & -l_{23} \\
    -l_{13} & -l_{23} & l_{22} + l_{11} - l_{33}
\end{pmatrix}
\]

Fedorov, Zenkov, Nonlinearity 2005

\[
\ell_d(W_k) = \frac{1}{2} \text{Trace}(W_k \mathbb{J}) \quad \text{(Moser-Veselov 1991)}
\]
Momentum locus

Fedorov, Zenkov, 2005

\[ \ell_d(W_k) = \frac{1}{2} \text{Trace}(W_k \mathbb{J}) \quad (\text{Moser-Veselov 1991}) \]

Momentum locus

\[ \mathcal{U} := \{ L^*_W \ell'_d(W) : W \in S \} \subset so_3^* \cong \mathbb{R}^3(\tilde{M}) \]

Components of \( \tilde{M} \) satisfy an algebraic equation of degree four.
Momentum locus

Fedorov, Zenkov, Nonlinearity 2005

\[ \ell_d(W_k) = \frac{1}{2} \text{Trace}(W_k \mathbb{J}) \quad \text{(Moser-Veselov 1991)} \]

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Components of \( \tilde{M} \) satisfy an algebraic equation of degree four.
Components of $\mathbf{M}$ satisfy an algebraic equation of degree four.

\[
\begin{pmatrix}
0 & -M_3 & M_2 \\
M_3 & 0 & -M_1 \\
-M_2 & M_1 & 0
\end{pmatrix} = \mathbf{WJ} - \mathbf{JW}^T.
\]
Components of $\tilde{M}$ satisfy an algebraic equation of degree four.
Discrete Dynamics

Discrete Euler-Poincaré-Suslov equations

\[
\vec{M}_{k+1} = W_k^T \vec{M}_k + \lambda_k \vec{e}_3
\]

\((\vec{M}_k = W_k \mathbb{J} - \mathbb{J} W_k^T, \ W_k \in S).\)
Discrete Dynamics

Discrete Euler-Poincaré-Suslov equations

\[ \tilde{M}_{k+1} = W_k^T \tilde{M}_k + \lambda_k \tilde{e}_3 \]

\[ (\tilde{M}_k = W_k \mathcal{J} - \mathcal{J} W_k^T, \quad W_k \in S). \]

Fedorov, Zenkov, \((S = \exp({\mathfrak{a}}))\)

Multivalued map \(\mathbb{RP}^2 \rightarrow \mathbb{RP}^2\).

System has 4 branches (2 real and 2 complex).
Discrete Dynamics

Discrete Euler-Poincaré-Suslov equations

\[ \tilde{M}_{k+1} = W_k^T \tilde{M}_k + \lambda_k \tilde{e}_3 \]

(\( \tilde{M}_k = W_k \mathbb{J} - \mathbb{J} W_k^T \), \( W_k \in S \)).

Fedorov, Zenkov, \((S = \exp(\mathfrak{d}))\)
Multivalued map \(\mathbb{RP}^2 \rightarrow \mathbb{RP}^2\).
System has 4 branches (2 real and 2 complex).

Our approach \((S = \text{Cayley}(\mathfrak{d}))\)
Multivalued map \(\mathbb{R}^2 \rightarrow \mathbb{R}^2\).
System has 4 branches (2 real and 2 complex).
Conservation of Energy

Constrained energy is preserved
Conservation of Energy

Constrained energy is preserved
Contour lines in $\mathbb{RP}^2 \ (S = \exp(\mathbf{d}))$
Conservation of Energy

Constrained energy is preserved
Contour lines in $\mathbb{R}^2 (S = \text{Cayley}(\partial))$
Conservation of Energy

Constrained energy is preserved

Contour lines in $\mathbb{R}^2$ ($\mathcal{S} = \text{Cayley}(\vartheta)$)

$$H_c(\Omega_1, \Omega_2) = \frac{1}{2} \left( l_{11} \Omega_1^2 + l_{22} \Omega_2^2 \right) \left( \frac{1 + \varepsilon^2 P_2(\Omega_1, \Omega_2)}{1 + \varepsilon^2 P_4(\Omega_1, \Omega_2)} \right)$$
Dynamics of Discrete Suslov Problem

Follow small real branch
Use our approach (Cayley Transformation)
Line of equilibria corresponding to $l_{13}\Omega_1 + l_{23}\Omega_2 = 0$. 
Dynamics of Discrete Suslov Problem

Follow small real branch
Use our approach (Cayley Transformation)
Line of equilibria corresponding to $l_{13}\Omega_1 + l_{23}\Omega_2 = 0$. 
Dynamics of Discrete Suslov Problem

Missing aspect:
• Explicit integration

\[ W_k = f(v_k), \quad v_k = k\Delta + v_0. \]
Alternative Discretization

Continuous data

\[ \mathcal{D}_g = L_{g^*} \mathcal{D}, \quad \mathcal{L}(g, \dot{g}) = \ell(\Omega) = \frac{1}{2} \text{Trace} \left( \hat{\Omega}(\hat{\Omega})^T J \right) \]
Alternative Discretization

Continuous data

\[ \mathcal{D}_g = L_{g \ast \partial}, \quad \mathcal{L}(g, \dot{g}) = \ell(\Omega) = \frac{1}{2} \text{Trace} \left( \hat{\Omega}(\hat{\Omega})^T \mathbb{I} \right) \]

Discretization proposed by Fedorov-Zenkov makes choices

\[ \mathcal{D}_d = \{ W_k \in \mathcal{S} = \exp(\partial) \} \]

\[ \mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k) = \frac{1}{2} \text{Trace}(W_k \mathbb{I}) \]
Alternative Discretization

Continuous data

\[ \mathcal{D}_g = L_{g^*} \mathcal{O}, \quad \mathcal{L}(g, \dot{g}) = \ell(\Omega) = \frac{1}{2} \text{Trace} \left( \hat{\Omega}(\hat{\Omega})^T \mathbb{J} \right) \]

Discretization proposed by Fedorov-Zenkov makes choices

\[ \mathcal{D}_d = \left\{ W_k \in S = \exp(\mathcal{O}) \right\} \]

\[ \mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k) = \frac{1}{2} \text{Trace}(W_k \mathbb{J}) \]

Not consistent!
Alternative Discretization

Continuous data

\[ \mathcal{D}_g = L_{g^*\mathcal{O}}, \quad \mathcal{L}(g, \dot{g}) = \ell(\Omega) = \frac{1}{2} \text{Trace} \left( \Omega (\Omega)^T J \right) \]

Discretization proposed by Fedorov-Zenkov makes choices

\[ \mathcal{D}_d = \{ W_k \in \mathcal{S} = \exp(\mathcal{O}) \} \]

\[ \mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k) = \frac{1}{2} \text{Trace}(W_k J) \]

Not consistent!

We propose new discretization

\[ \mathcal{D}_d = \{ W_k \in \mathcal{S} = \text{Cayley}(\mathcal{O}) \} \]

\[ \mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k) \]

\[ = \frac{1}{2\varepsilon^2} \text{Trace}(\text{Cayley}^{-1}(W_k) \text{Cayley}^{-1}(W_k)^T J) \]
Momentum locus

\[ \ell_d(W_k) = \frac{1}{2\varepsilon^2} \text{Trace}(\text{Cayley}^{-1}(W_k)\text{Cayley}^{-1}(W_k)^T J) \]

Momentum locus

\[ \mathcal{U} := \{ L^*_W \ell'_d(W) : W \in S \} \subset \mathfrak{so}(3)^* \cong \mathbb{R}^3(\tilde{M}) \]

\( S = \text{Cayley}(\partial) \)
Momentum locus

$$\mathcal{U} := \{ L^*_{\mathcal{W}} \ell'_d(\mathcal{W}) : \mathcal{W} \in S \} \subset \mathfrak{so}(3)^* \cong \mathbb{R}^3(\mathbf{\tilde{M}})$$

$$S = \text{Cayley}(\mathfrak{d})$$

$$M_1 = l_{11}\Omega_1 + \frac{1}{2}\Omega_2(l_{13}\Omega_1 + l_{23}\Omega_2) + \frac{1}{4}\Omega_1(l_{11}\Omega_1^2 + l_{22}\Omega_2^2)$$

$$M_2 = l_{22}\Omega_2 - \frac{1}{2}\Omega_1(l_{13}\Omega_1 + l_{23}\Omega_2) + \frac{1}{4}\Omega_2(l_{11}\Omega_1^2 + l_{22}\Omega_2^2)$$

$$M_3 = l_{13}\Omega_1 + l_{23}\Omega_2 + \frac{1}{2}(l_{11} - l_{22})\Omega_1\Omega_2$$

Components of $\mathbf{\tilde{M}}$ satisfy an algebraic equation of degree 7.
Discrete Dynamics

Discrete Euler-Poincaré-Suslov equations

\[ \hat{M}_{k+1} = W_k^T \hat{M}_k + \lambda_k \bar{e}_3 \]

\[(\hat{M}_k = L_{W_k}^* \ell'_d(W_k), \quad W_k \in S).\]
Discrete Dynamics

Discrete Euler-Poincaré-Suslov equations

\[ \tilde{M}_{k+1} = W_k^T \tilde{M}_k + \lambda_k \tilde{e}_3 \]

\[ (\hat{M}_k = L^*_{W_k} \ell'_d(W_k), \quad W_k \in S). \]

Multivalued map \( \mathbb{C}^2 \rightarrow \mathbb{C}^2 \).

System has 7 branches (1 real and 6 complex).
Conservation of Energy

if \( I_{11} = I_{22} \) Constrained energy is preserved.
Conservation of Energy

if $I_{11} = I_{22}$ Constrained energy is preserved.

$$H_c(\Omega_1, \Omega_2) = \frac{I_{11}}{2} \left( \Omega_1^2 + \Omega_2^2 \right) \left( 1 + \varepsilon^2 P_2(\Omega_1, \Omega_2) + \varepsilon^4 P_4(\Omega_1, \Omega_2) \right).$$
Conservation of Energy

if $I_{11} = I_{22}$ Constrained energy is preserved.

$$H_c(\Omega_1, \Omega_2) = \frac{I_{11}}{2} \left( \Omega_1^2 + \Omega_2^2 \right) \left( 1 + \epsilon^2 P_2(\Omega_1, \Omega_2) + \epsilon^4 P_4(\Omega_1, \Omega_2) \right).$$

Integral for general case: Current work.
Simulations - Small time step
Simulations - Small time step

Compare with continuous system
Simulations - Small time step

Compare with Fedorov-Zenkov discretization
Simulations - Small time step

Simultaneous comparison
Simulations - Large time step

\[ \omega_1, \omega_2 \]
Simulations - Large time step

Compare with continuous system
Simulations - Large time step

Compare with Fedorov-Zenkov discretization
Simulations - Large time step

Simultaneous comparison
Asymptotics of conserved quantity (if it exists!)
Experiments on explicit integrability for the case $l_{11} = l_{22}$

Conserved energy

$$E = \frac{1}{2} (M_1^2 + M_2^2)$$
Experiments on explicit integrability for the case $l_{11} = l_{22}$

Conserved energy

$$ E = \frac{1}{2} (M_1^2 + M_2^2) $$

Parametrize level set

$$ M_1(t) = \sqrt{\frac{E}{2(l_{13}^2 + l_{23}^2)}} (-l_{23} \tanh(t) + l_{13} \text{sech}(t)) $$

$$ M_2(t) = \sqrt{\frac{E}{2(l_{13}^2 + l_{23}^2)}} (l_{13} \tanh(t) + l_{23} \text{sech}(t)) $$
Experiments on explicit integrability for the case $l_{11} = l_{22}$

Conserved energy

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$$M_1^{(k)} = M_1(t_k), \quad M_2^{(k)} = M_2(t_k)$$
Experiments on explicit integrability for the case $l_{11} = l_{22}$

Conserved energy

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$$M_1^{(k)} = M_1(t_k), \quad M_2^{(k)} = M_2(t_k)$$

$$\Delta_k = t_{k+1} - t_k, \quad \text{Constant?}$$
Experiments on explicit integrability for the case $I_{11} = I_{22}$

$\Delta_1 = 0.1611957340$
$\Delta_2 = 0.1610722014$
$\Delta_3 = 0.1609643223$
$\Delta_4 = 0.1608883641$
$\Delta_5 = 0.1608576431$

\[ \vdots \]

$\Delta_{80} = 0.16130742$
Thanks!