# Discrete nonholonomic mechanics on Lie groups

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$\mathcal{D}_q \subset T_qQ$ is the annihilator of the one-forms on $Q$:

$$\beta^j = \sum_{s=1}^{n} \beta_s^i(q)dq^s, \qquad j = 1, \ldots, m.$$

# Equations of motion (Lagrangian formulation)

Lagrange-D'Alembert principle (Newton's Law):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \underbrace{\lambda_1 \beta^1(q) + \cdots + \lambda_m \beta^m(q)}_{\text{Reaction Forces}}.$$

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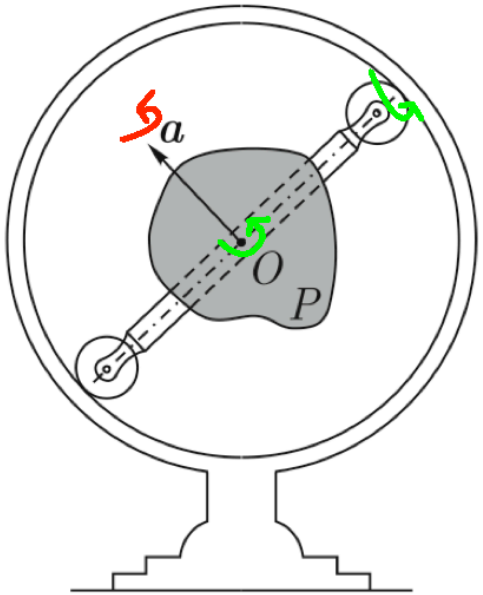


Suslov Problem (1902) $\quad \mathbf{\Omega} \cdot \mathbf{a} = 0.$

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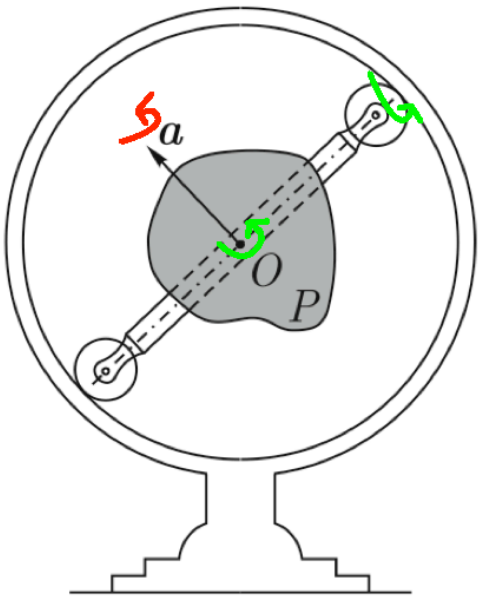


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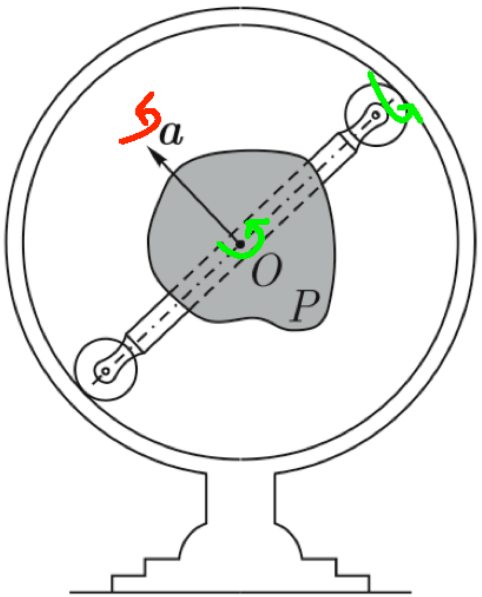


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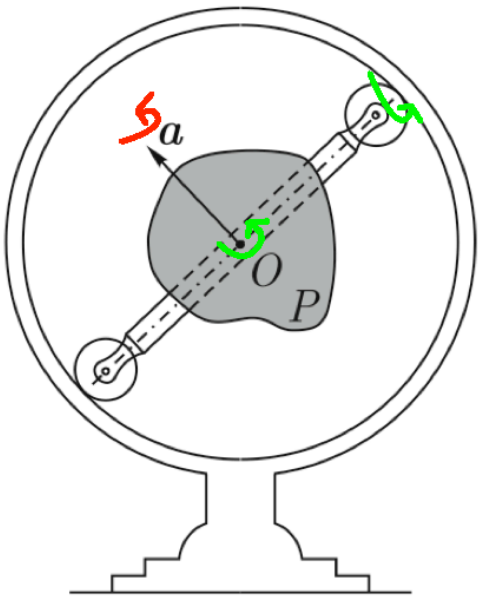


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Equations are *not* variational in the usual sense.

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Does not preserve energy in general.

# How to select discrete data?

Given $(\mathcal{D}, \mathcal{L})$ select $(\mathcal{D}_d, \mathcal{L}_d)$.

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$$\dot{\mathbf{M}} = \mathrm{ad}^*_{\mathbf{\Omega}}\mathbf{M} + \sum_{j=1}^{m} \lambda_j a^j$$

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$\langle \mathbf{M}, \mathbb{I}^{-1} a^j\rangle = 0, \qquad \mathbf{M} \in \mathbb{I}(\mathfrak{d}) = \mathfrak{u} \subset \mathfrak{g}^*$

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- **Key Observation:** Euler-Poincaré-Suslov equations are defined in a linear subspace of $\mathfrak{g}^*$.

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Fedorov, Zenkov, 2005

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Fedorov-Zenkov: $\mathcal{S} = \exp(\mathfrak{d})$

Our investigation: $\mathcal{S} = \mathrm{Cayley}(\mathfrak{d})$

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# Discrete Euler-Poincaré-Suslov equations

Discrete Lagrange-D'Alembert Principle

$$D_1\mathcal{L}_d(g_k, g_{k+1}) + D_2\mathcal{L}_d(g_{k-1}, g_k) = \sum_{j=1}^{m} \lambda_j^{(k)} L^*_{g_k^{-1}} a^j$$

Pull-it back to $\mathfrak{g}^*$ by left translation (Bobenko-Suris 1999).

$$-\operatorname{Ad}^*_{W_k^{-1}} M_{k+1} + M_k = \sum_{j=1}^{m} \lambda_j^{(k)} a^j$$

$$M_k = L^*_{W_{k-1}} \ell'_d(W_{k-1}) \in \mathfrak{g}^*$$

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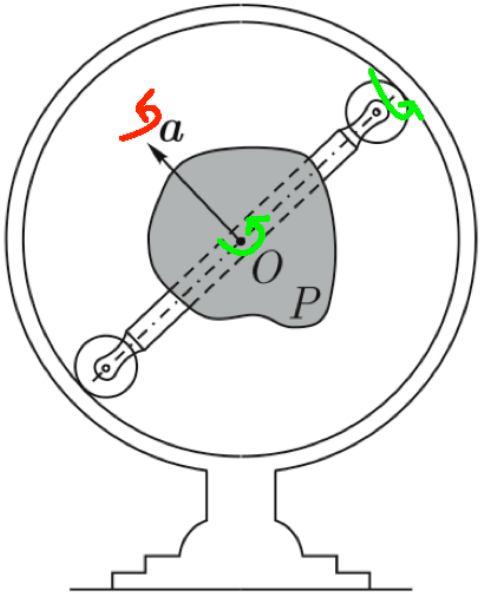
- **Key Observation:** Discrete Euler-Poincaré-Suslov equations are defined in the *Momentum Locus* $\mathcal{U} \subset \mathfrak{g}^*$ - a nonlinear subset of $\mathfrak{g}^*$.

# Outline

# Suslov Problem




$$Q = G = SO(3)$$

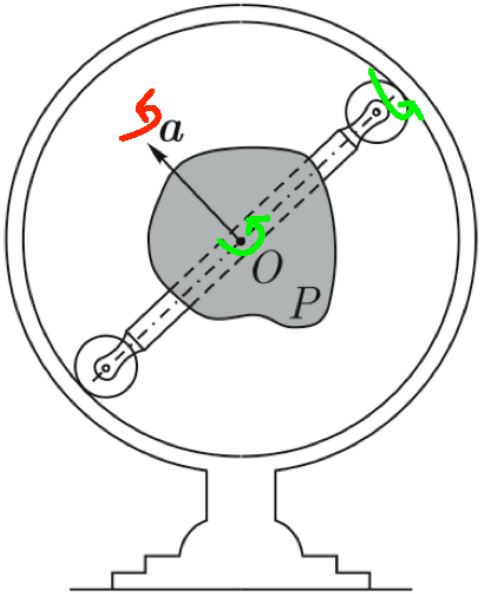
Nonholonomic constraint:

$$\mathbf{\Omega} \cdot \mathbf{a} = 0,$$

Lagrangian:

$$\mathcal{L}(g, \dot{g}) = \ell(\mathbf{\Omega}) = \frac{1}{2}\mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Omega}$$

# Suslov Problem




$$Q = G = SO(3)$$

Select body frame:

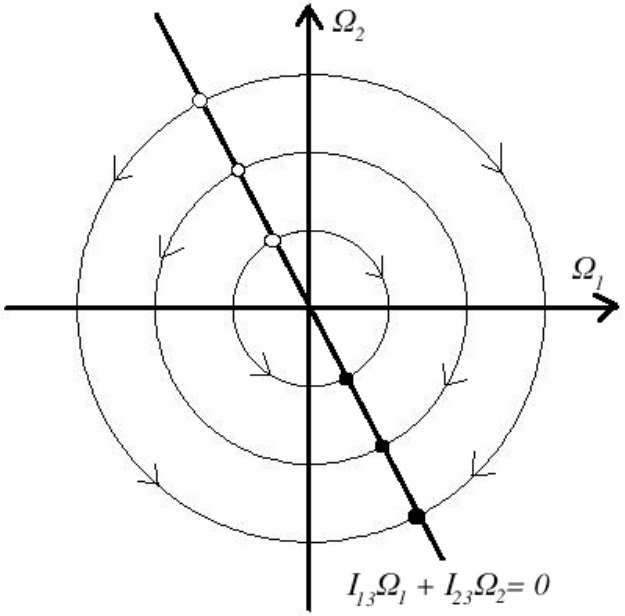
$$\mathbf{a} = \mathbf{e_3}, \qquad \mathbb{I} = \begin{pmatrix} I_{11} & 0 & I_{13} \\ 0 & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}$$

$$\Omega_3 = 0, \qquad \ell(\mathbf{\Omega}) = \frac{1}{2}\mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Omega}$$

# Suslov Problem - Reduced Dynamics

Reduced equations:

$$\frac{d}{dt}\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} \frac{-\Omega_2(I_{13}\Omega_1+I_{23}\Omega_2)}{I_{11}} \\ \frac{\Omega_1(I_{13}\Omega_1+I_{23}\Omega_2)}{I_{22}} \end{pmatrix}$$




Key aspects: Integrable, Hamiltonian, asymptotic behavior, no global preserved measure.

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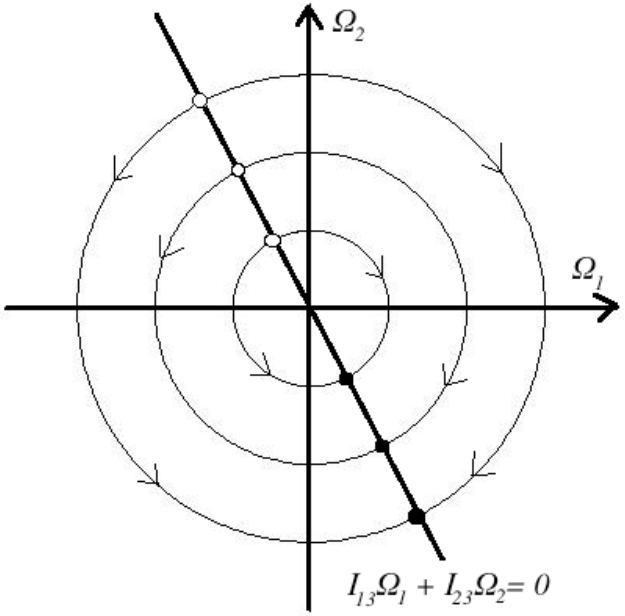


Key aspects: Integrable, Hamiltonian, asymptotic behavior, no global preserved measure.

Dragović-Gajić 2008- Hirota-Kimura scheme. Explicit, integrable.

# Discrete Suslov Problem

Fedorov, Zenkov, 2005

Discrete constraint space

$$W_k \in \mathcal{S} = \exp(\mathfrak{d})$$

$$\mathfrak{d} = \{\boldsymbol{\Omega} \in \mathbb{R}^3 \,:\, \Omega_3 = 0\}$$

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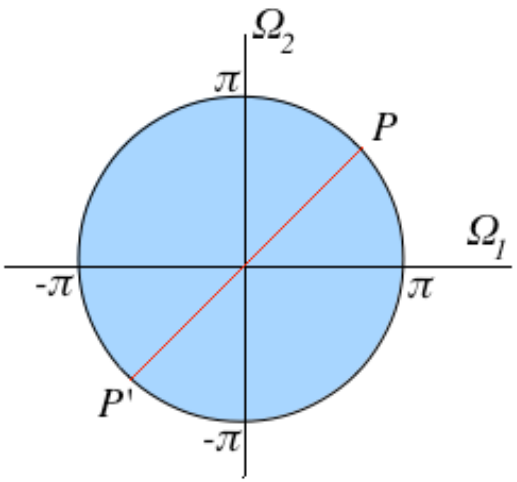


$$\mathcal{S} \cong \mathbb{RP}^2 \subset SO(3).$$

# Discrete Suslov Problem

## Our approach

Discrete constraint space

$$W_k \in \mathcal{S} = \text{Cayley}(\mathfrak{d})$$

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$$\text{Cayley} : \mathbb{R}^3 \cong \mathfrak{so}(3) \to SO(3)$$
$$\mathbf{\Omega} \cong \widehat{\mathbf{\Omega}} \mapsto \left(E - \frac{1}{2}\widehat{\mathbf{\Omega}}\right)\left(E + \frac{1}{2}\widehat{\mathbf{\Omega}}\right)^{-1}$$

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$$\text{Cayley}(\mathbf{\Omega}) = \text{Rotation with axis } \mathbf{\Omega} \text{ and angle } \theta$$
$$\cos\theta = \frac{4 - ||\mathbf{\Omega}||^2}{4 + ||\mathbf{\Omega}||^2}, \qquad 0 \leq \theta < \pi.$$

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Inverse Cayley Transformation

$$\text{Cayley}^{-1} : SO(3) \setminus \mathcal{R}_\pi \to \mathfrak{so}(3) \cong \mathbb{R}^3$$
$$W \mapsto 2(W - E)(W + E)^{-1}.$$

# Discrete Suslov Problem

## Our approach

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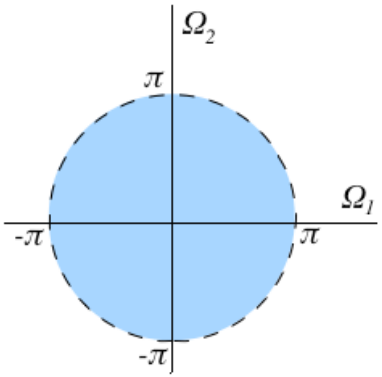


$$W = \begin{pmatrix} \frac{4+\Omega_1^2-\Omega_2^2}{4+\Omega_1^2+\Omega_2^2} & \frac{2\Omega_1\Omega_2}{4+\Omega_1^2+\Omega_2^2} & \frac{4\Omega_2}{4+\Omega_1^2+\Omega_2^2} \\ \frac{2\Omega_1\Omega_2}{4+\Omega_1^2+\Omega_2^2} & \frac{4-\Omega_1^2+\Omega_2^2}{4+\Omega_1^2+\Omega_2^2} & \frac{-4\Omega_1}{4+\Omega_1^2+\Omega_2^2} \\ \frac{-4\Omega_2}{4+\Omega_1^2+\Omega_2^2} & \frac{4\Omega_1}{4+\Omega_1^2+\Omega_2^2} & \frac{4-\Omega_1^2-\Omega_2^2}{4+\Omega_1^2+\Omega_2^2} \end{pmatrix}$$

$\mathbb{R}^2 \cong \mathfrak{d} \cong \mathcal{S} \subset SO(3)$.

# Choice of Discrete Lagrangian

$$\ell(\mathbf{\Omega}) = \frac{1}{2}\langle \mathbb{I}\mathbf{\Omega}, \mathbf{\Omega}\rangle = \frac{1}{2}\mathsf{Trace}\left(\widehat{\mathbf{\Omega}}(\widehat{\mathbf{\Omega}})^T\mathbb{J}\right) = \ell(\widehat{\mathbf{\Omega}})$$

$$\mathbb{J} = \frac{1}{2}\begin{pmatrix} I_{22}+I_{33}-I_{11} & 0 & -I_{13} \\ 0 & I_{11}+I_{33}-I_{22} & -I_{23} \\ -I_{13} & -I_{23} & I_{22}+I_{11}-I_{33} \end{pmatrix}$$

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Fedorov, Zenkov, Nonlinearity 2005

$$\ell_d(W_k) = \frac{1}{2}\mathrm{Trace}(W_k\mathbb{J}) \qquad \text{(Moser-Veselov 1991)}$$

# Momentum locus

Fedorov, Zenkov, 2005

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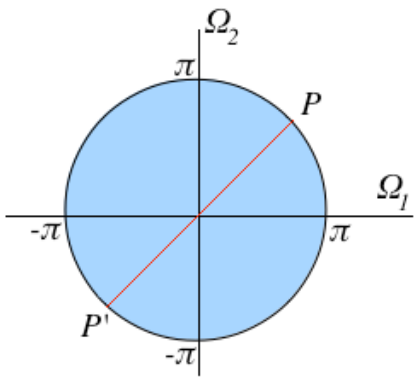
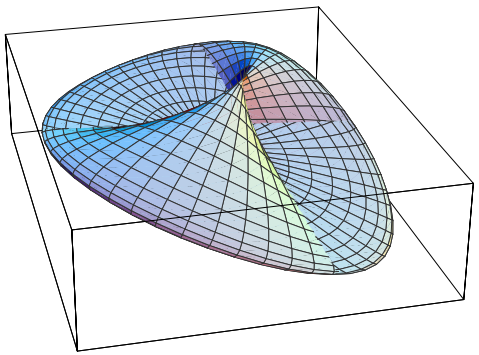
Momentum locus

$$\mathcal{U} := \left\{ L_W^* \ell_d'(W) \,:\, W \in \mathcal{S} \right\} \subset \mathfrak{so}(3)^* \cong \mathbb{R}^3(\vec{M})$$




$\mathcal{S} = \exp(\mathfrak{d})$

$\longmapsto$



Components of $\vec{M}$ satisfy an algebraic equation of degree four.

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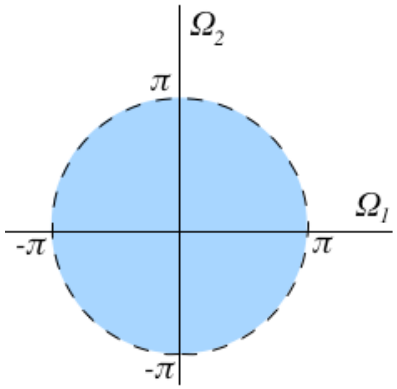


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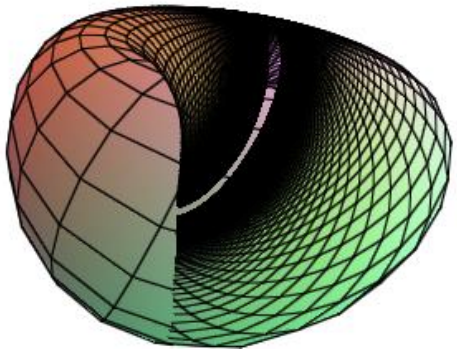

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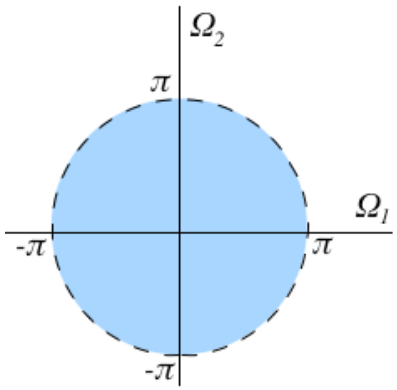


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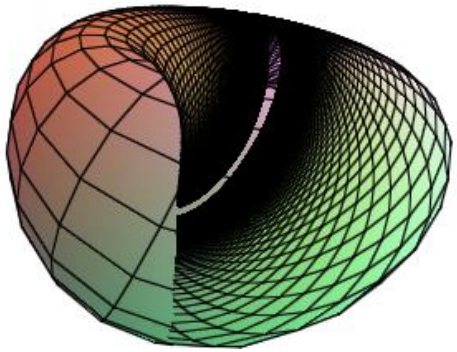

$$\begin{pmatrix} 0 & -M_3 & M_2 \\ M_3 & 0 & -M_1 \\ -M_2 & M_1 & 0 \end{pmatrix} = W\mathbb{J} - \mathbb{J}W^T.$$

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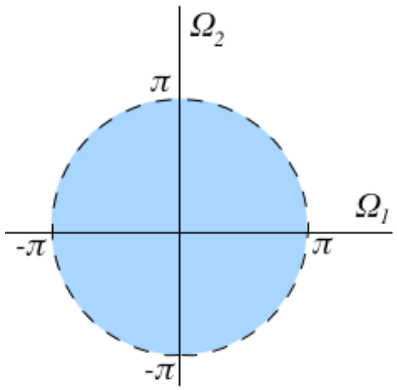


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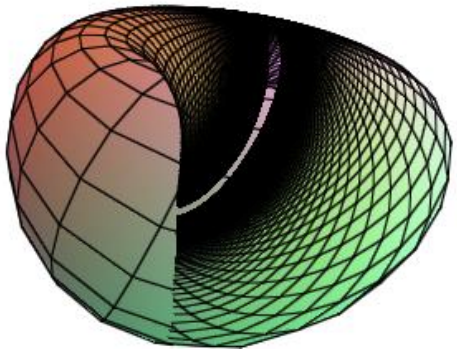

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$$M_1 = \frac{4I_{11}\Omega_1 + 2I_{13}\Omega_1\Omega_2 + 2I_{23}\Omega_2^2}{4 + \Omega_1^2 + \Omega_2^2}, \qquad M_2 = \ldots$$

Components of $\vec{M}$ satisfy an algebraic equation of degree four.

# Discrete Dynamics

Discrete Euler-Poincaré-Suslov equations

$$\vec{M}_{k+1} = W_k^T \vec{M}_k + \lambda_k \vec{e}_3$$
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Multivalued map $\mathbb{RP}^2 \to \mathbb{RP}^2$.

System has 4 branches (2 real and 2 complex).

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Our approach ($\mathcal{S} = \mathrm{Cayley}(\mathfrak{d})$)

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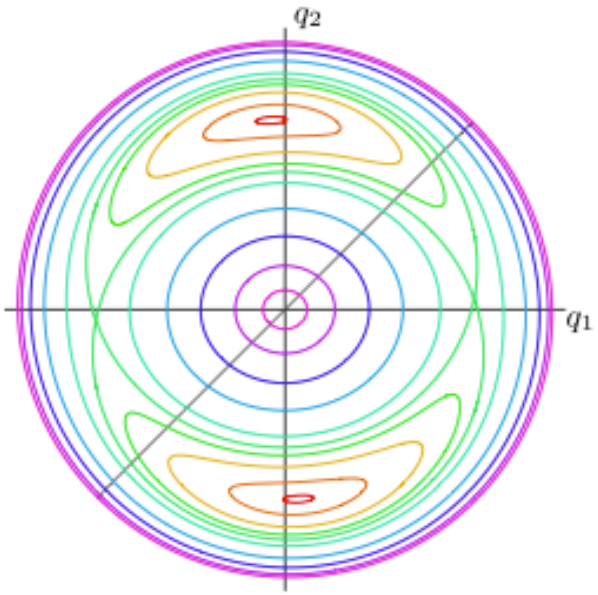
Contour lines in $\mathbb{RP}^2$ ($\mathcal{S} = \exp(\mathfrak{d})$)



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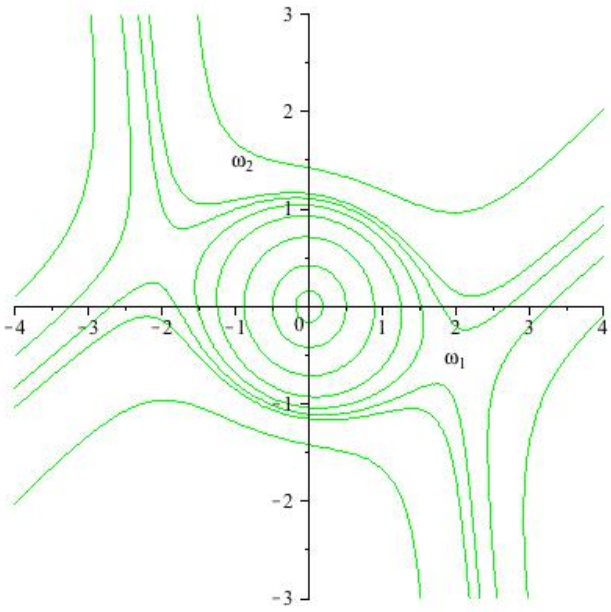

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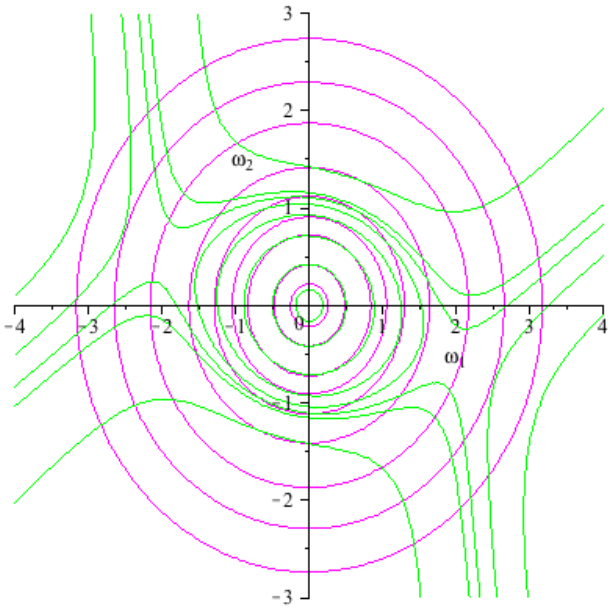


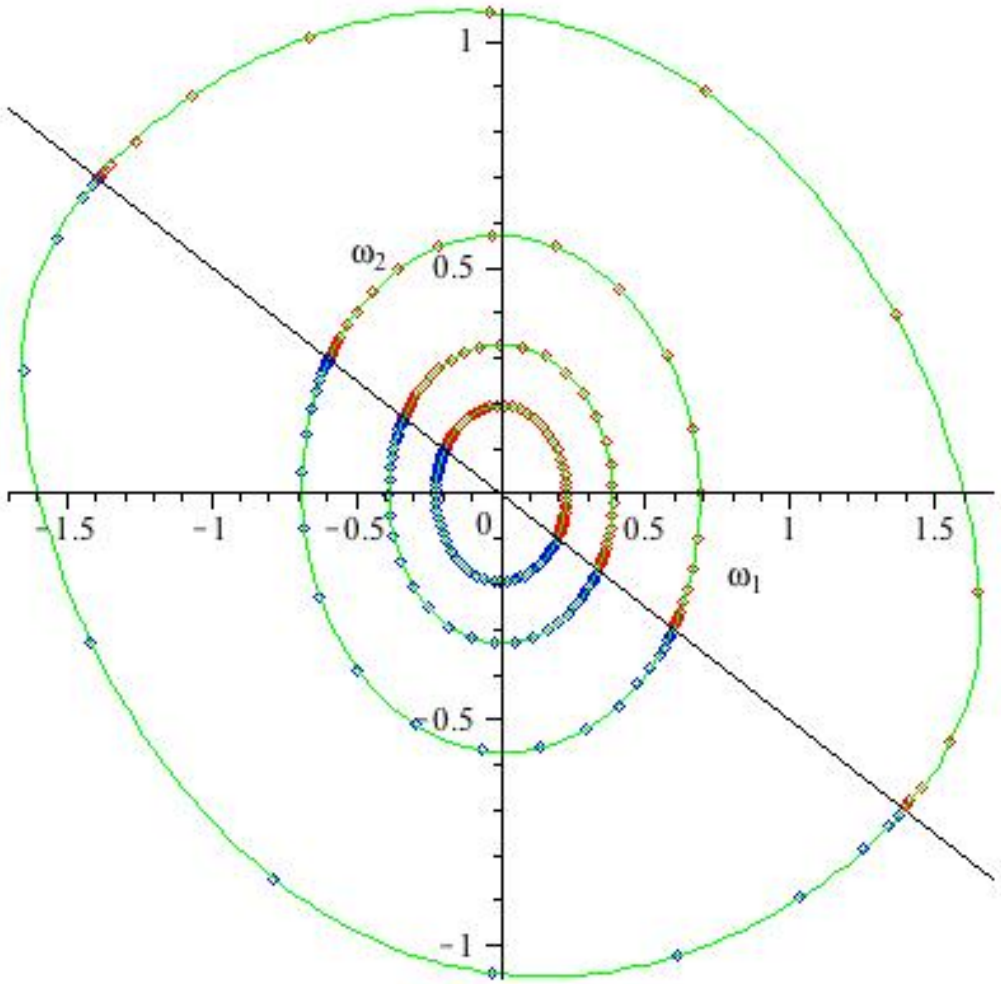

$$H_c(\Omega_1, \Omega_2) = \frac{1}{2}\left(I_{11}\Omega_1^2 + I_{22}\Omega_2^2\right)\left(\frac{1 + \varepsilon^2 P_2(\Omega_1, \Omega_2)}{1 + \varepsilon^2 P_4(\Omega_1, \Omega_2)}\right)$$

# Dynamics of Discrete Suslov Problem

Follow small real branch

Use our approach (Cayley Transformation)

Line of equilibria corresponding to $I_{13}\Omega_1 + I_{23}\Omega_2 = 0$.



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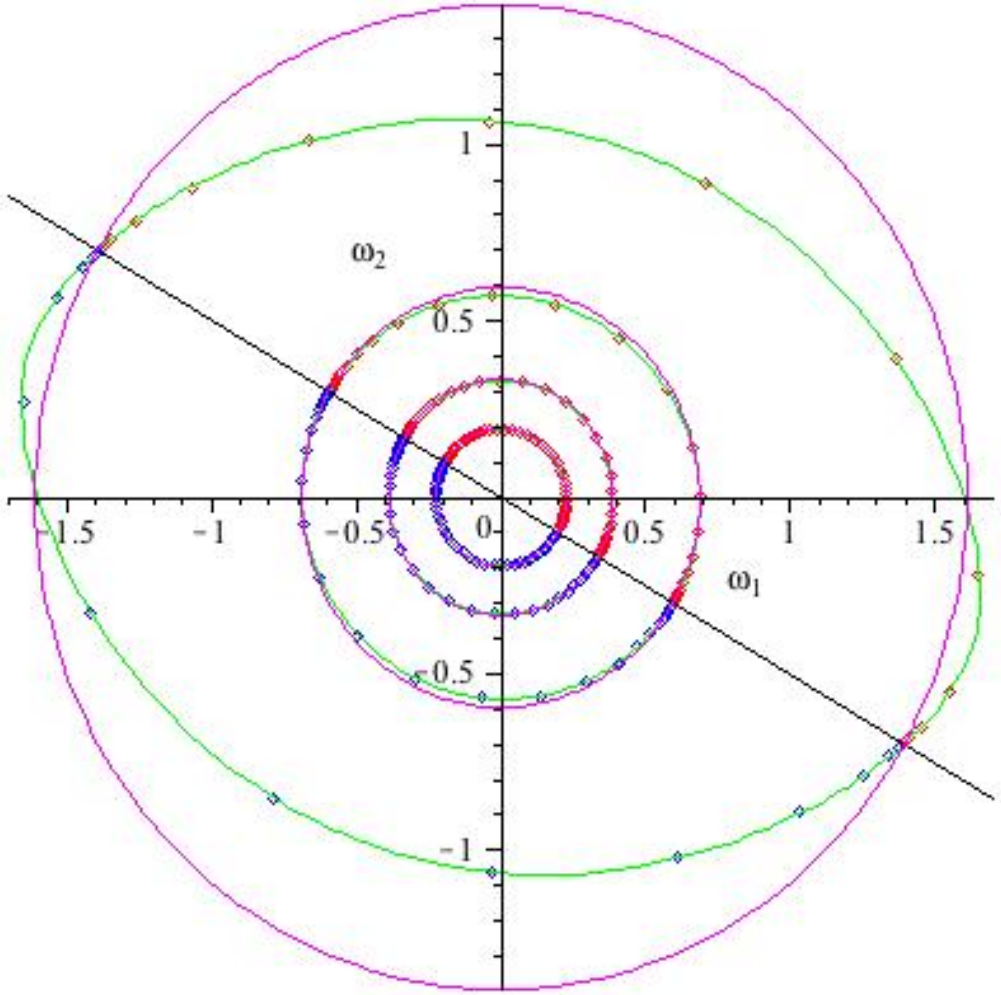

# Dynamics of Discrete Suslov Problem

Missing aspect:

- Explicit integration

$$W_k = f(v_k), \qquad v_k = k\Delta + v_0.$$

# Alternative Discretization

Continuous data

$$\mathcal{D}_g = L_{g*}\mathfrak{d}, \qquad \mathcal{L}(g, \dot{g}) = \ell(\mathbf{\Omega}) = \frac{1}{2}\mathrm{Trace}\left(\widehat{\mathbf{\Omega}}(\widehat{\mathbf{\Omega}})^T \mathbb{J}\right)$$

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$$\mathcal{D}_d = \{W_k \in \mathcal{S} = \exp(\mathfrak{d})\}$$

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Not consistent!

We propose new discretization

$$\mathcal{D}_d = \{W_k \in \mathcal{S} = \text{Cayley}(\mathfrak{d})\}$$
$$\mathcal{L}_d(g_k, g_{k+1}) = \ell_d(W_k)$$
$$= \frac{1}{2\varepsilon^2}\text{Trace}(\text{Cayley}^{-1}(W_k)\text{Cayley}^{-1}(W_k)^T\mathbb{J})$$

# Momentum locus

$$\ell_d(W_k) = \frac{1}{2\varepsilon^2}\mathrm{Trace}(\mathrm{Cayley}^{-1}(W_k)\mathrm{Cayley}^{-1}(W_k)^T\mathbb{J})$$

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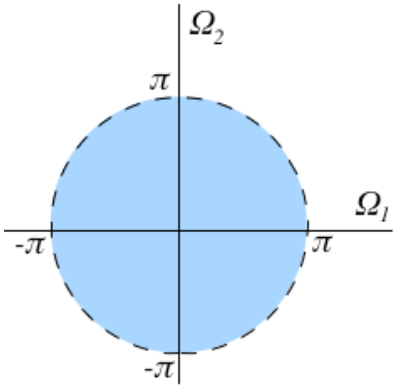


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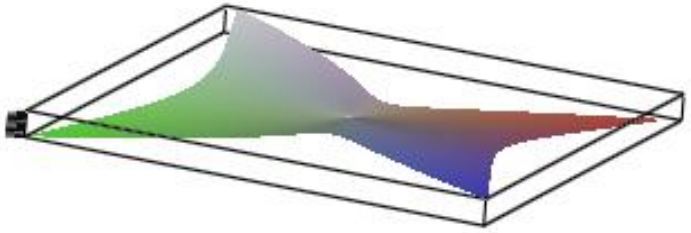
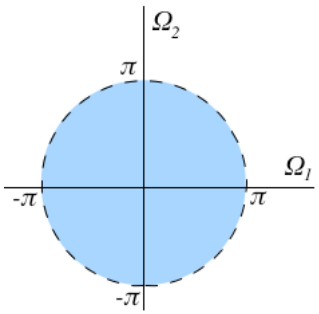


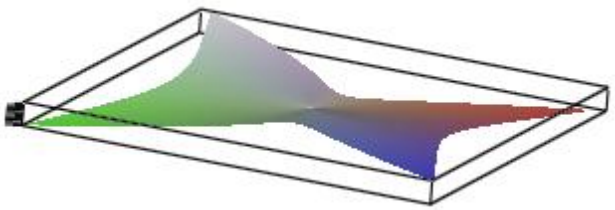
# Momentum locus

$$\mathcal{U} := \{L_W^* \ell_d'(W) \,:\, W \in \mathcal{S}\} \subset \mathfrak{so}(3)^* \cong \mathbb{R}^3(\vec{M})$$




$\mathcal{S} = \text{Cayley}(\mathfrak{d})$

$\mapsto$



$$M_1 = I_{11}\Omega_1 + \frac{1}{2}\Omega_2(I_{13}\Omega_1 + I_{23}\Omega_2) + \frac{1}{4}\Omega_1(I_{11}\Omega_1^2 + I_{22}\Omega_2^2)$$

$$M_2 = I_{22}\Omega_2 - \frac{1}{2}\Omega_1(I_{13}\Omega_1 + I_{23}\Omega_2) + \frac{1}{4}\Omega_2(I_{11}\Omega_1^2 + I_{22}\Omega_2^2)$$

$$M_3 = I_{13}\Omega_1 + I_{23}\Omega_2 + \frac{1}{2}(I_{11} - I_{22})\Omega_1\Omega_2$$

Components of $\vec{M}$ satisfy an algebraic equation of degree 7.

# Discrete Dynamics

Discrete Euler-Poincaré-Suslov equations

$$\vec{M}_{k+1} = W_k^T \vec{M}_k + \lambda_k \vec{e}_3$$
$$(\hat{M}_k = L^*_{W_k} \ell'_d(W_k), \quad W_k \in \mathcal{S}).$$

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Multivalued map $\mathbb{C}^2 \to \mathbb{C}^2$.

System has 7 branches (1 real and 6 complex).

# Conservation of Energy

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$$H_c(\Omega_1, \Omega_2) = \frac{I_{11}}{2}\left(\Omega_1^2 + \Omega_2^2\right)\left(1 + \varepsilon^2 P_2(\Omega_1, \Omega_2) + \varepsilon^4 P_4(\Omega_1, \Omega_2)\right).$$

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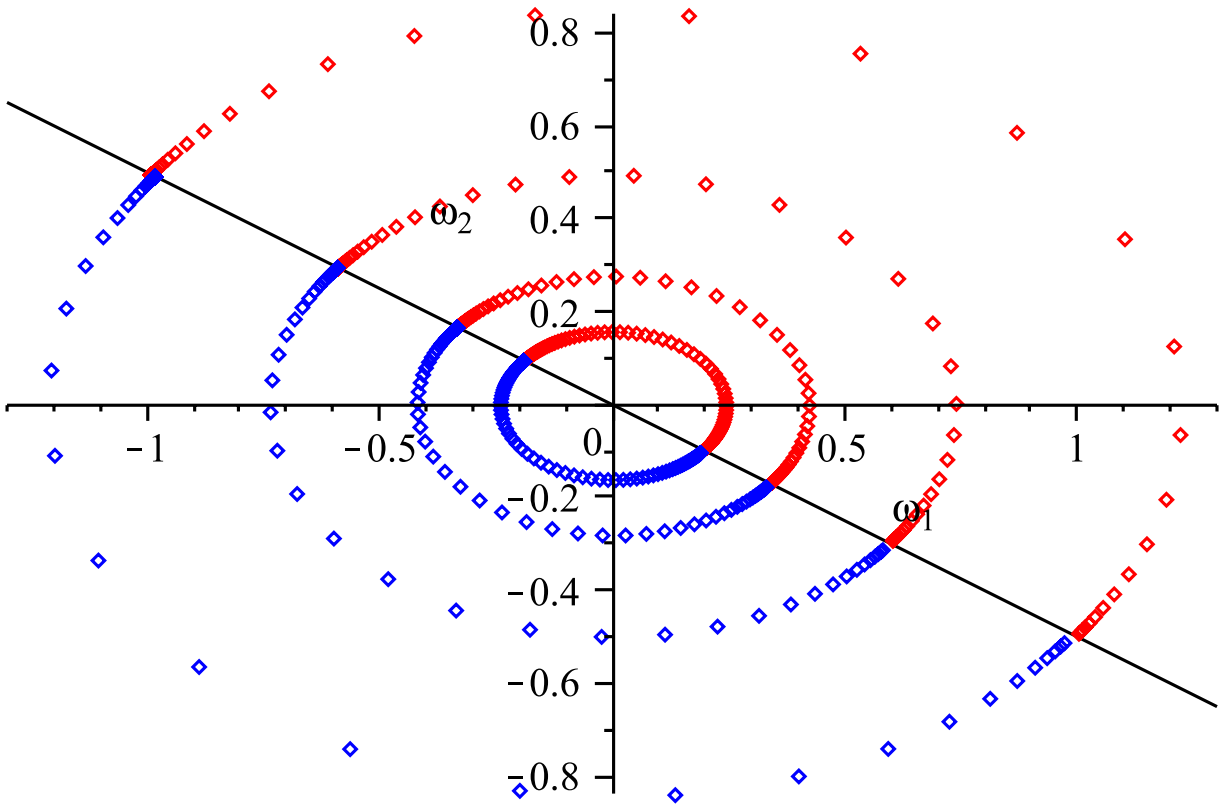
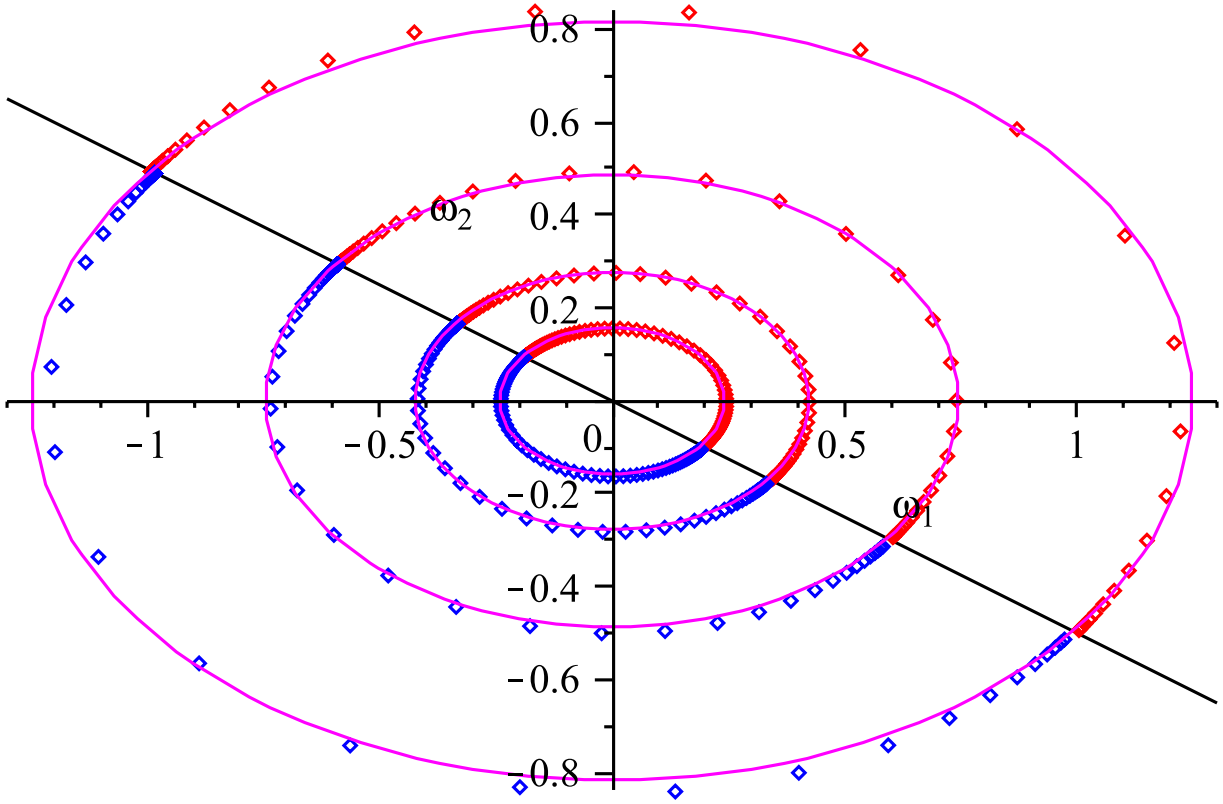
Integral for general case: Current work.

# Simulations - Small time step



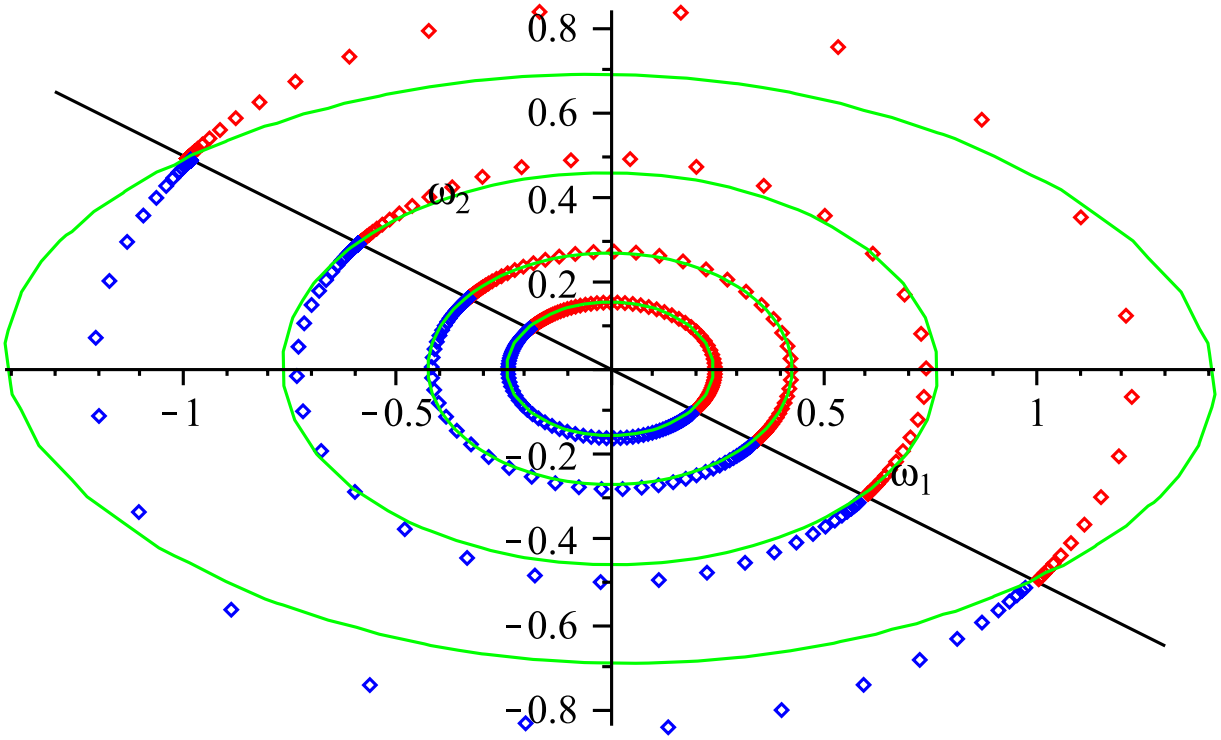
# Simulations - Small time step

Compare with continuous system



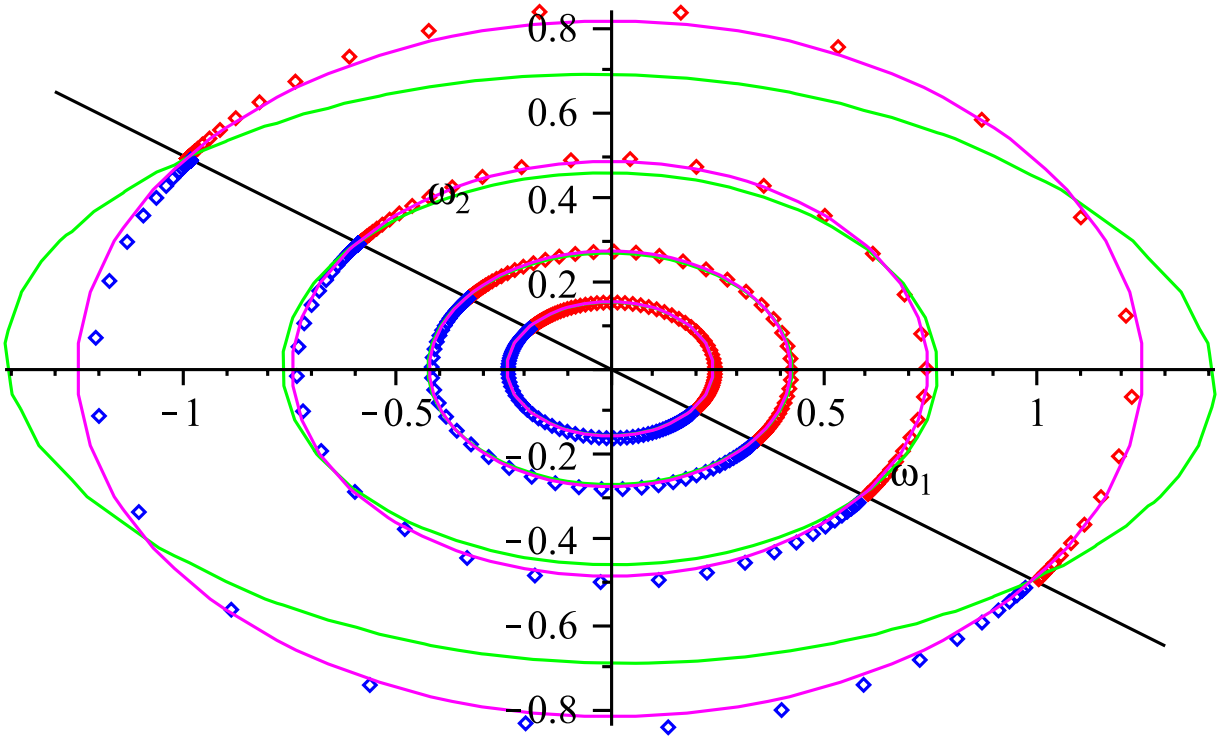
# Simulations - Small time step

Compare with Fedorov-Zenkov discretization

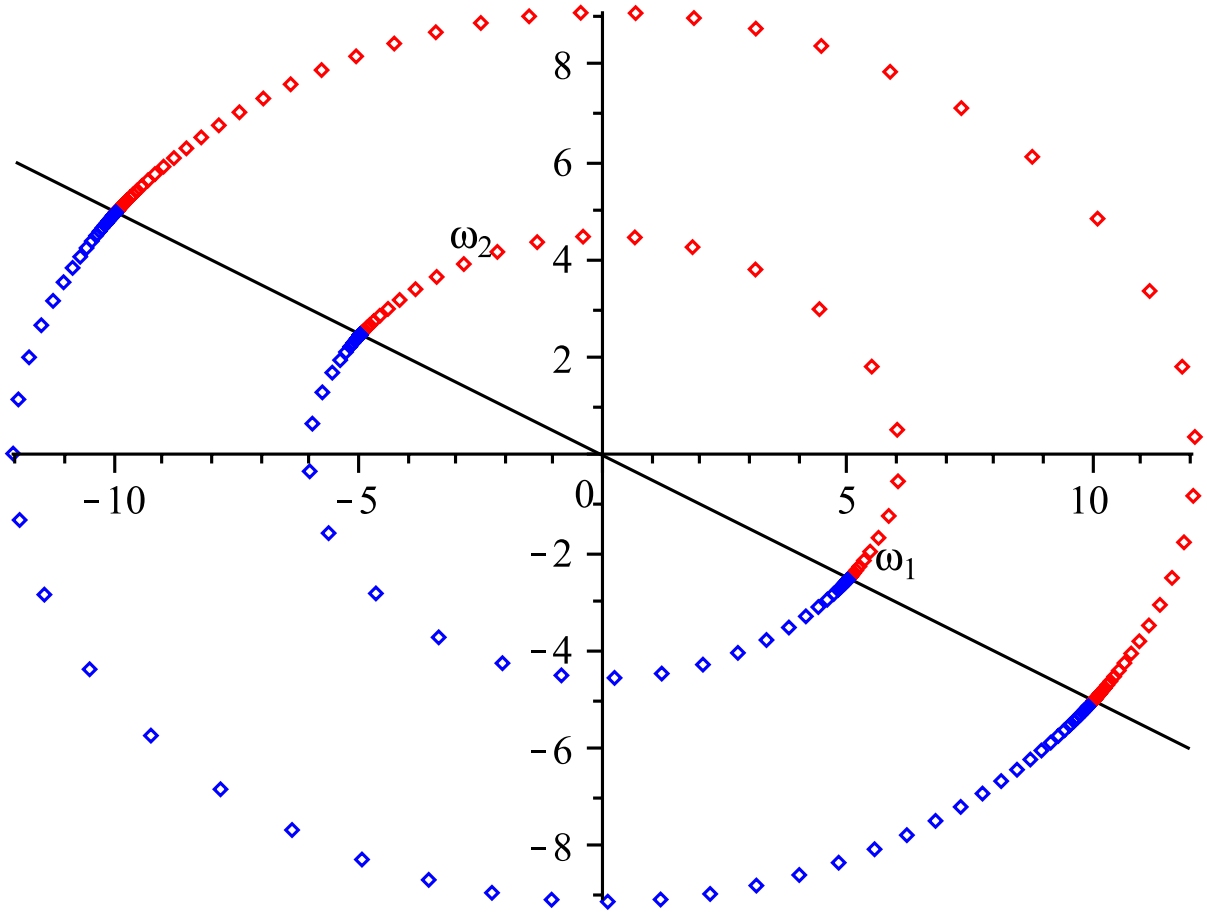


# Simulations - Small time step

Simultaneous comparison



# Simulations - Large time step



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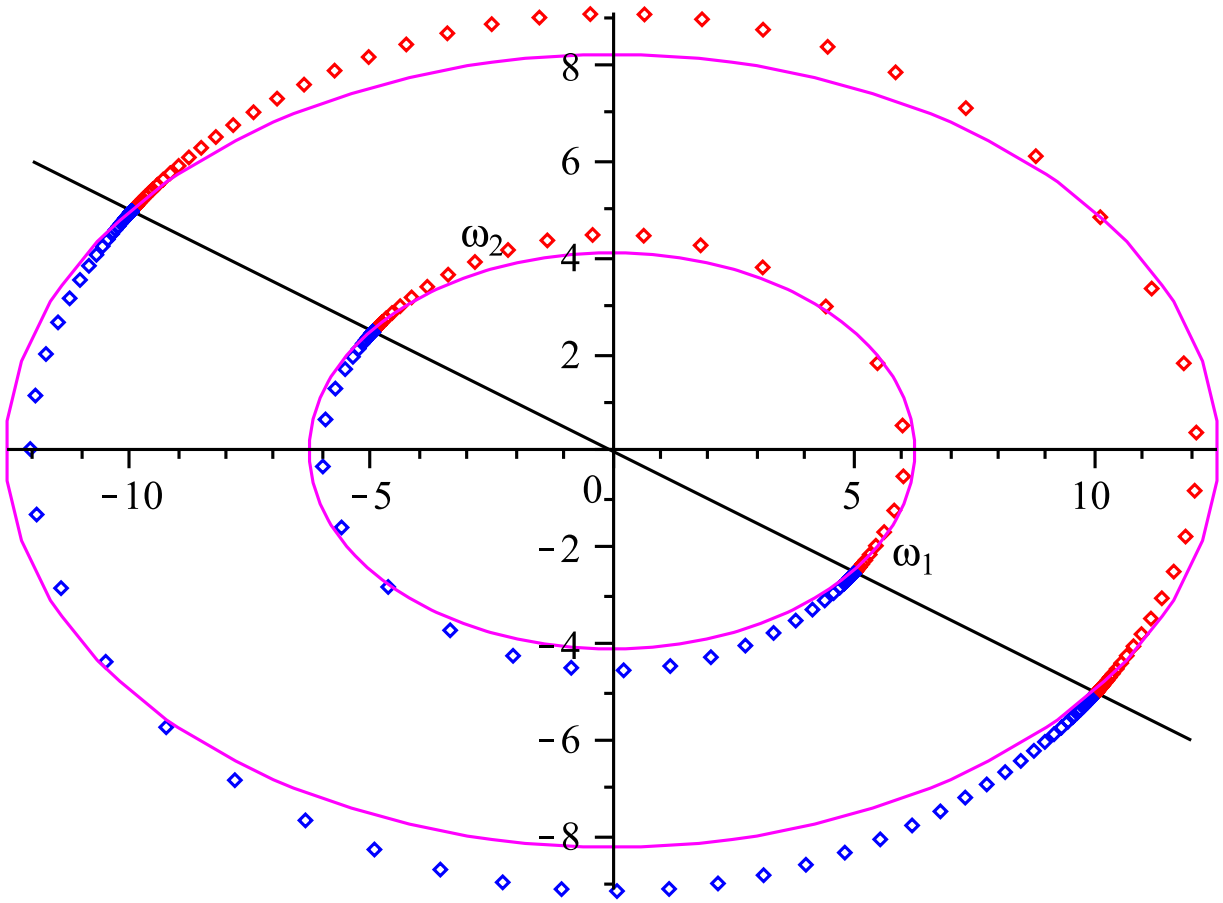

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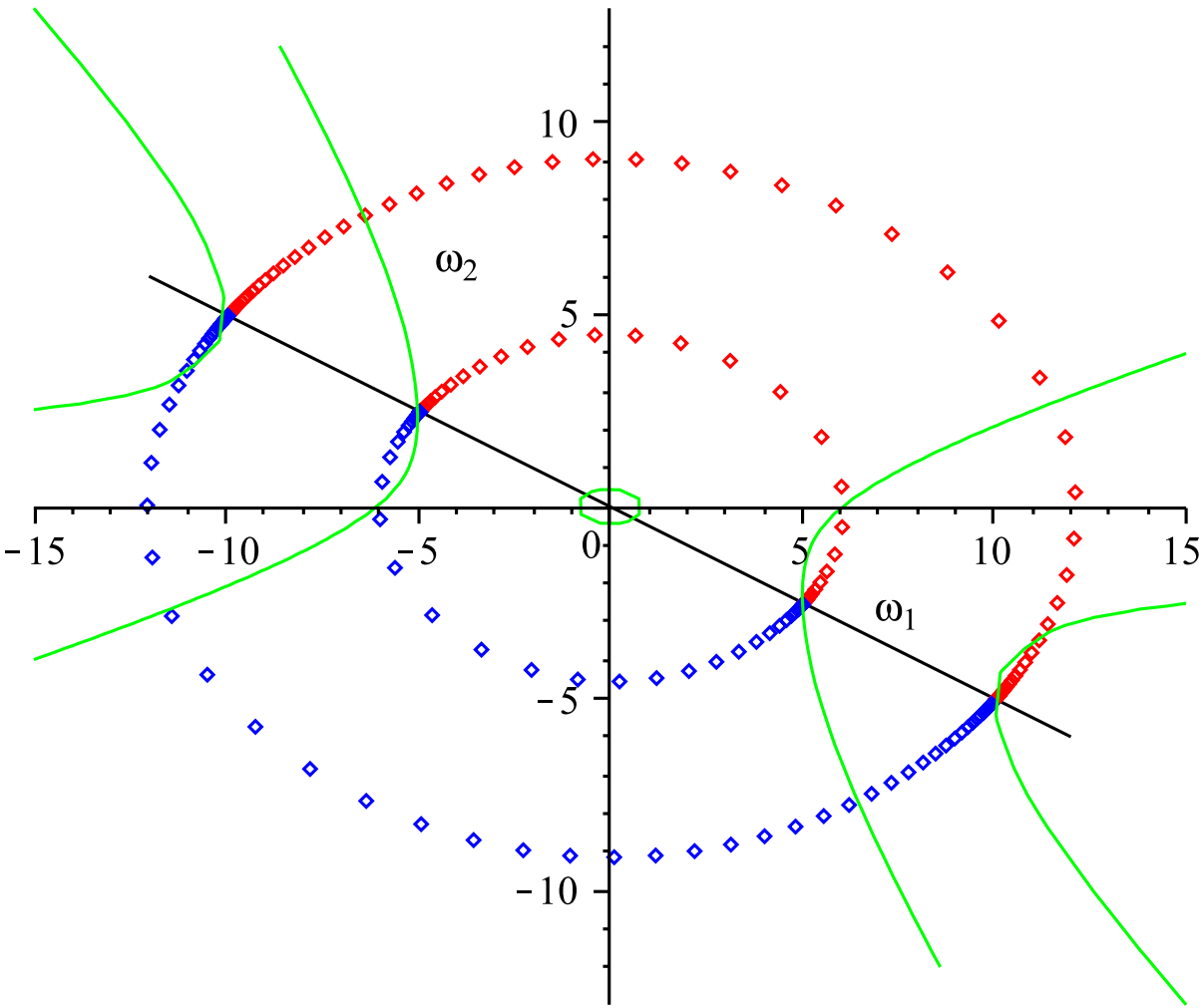

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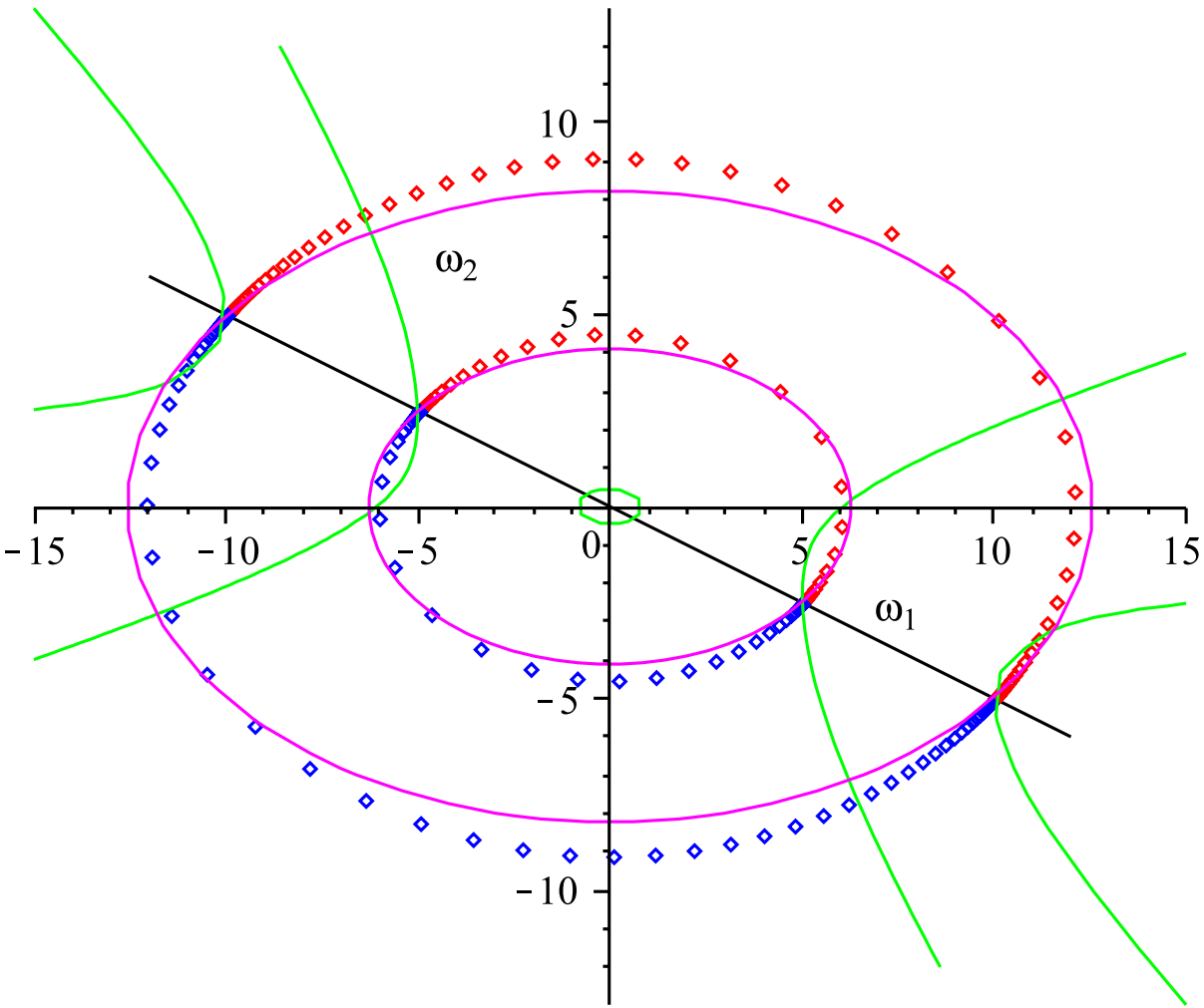
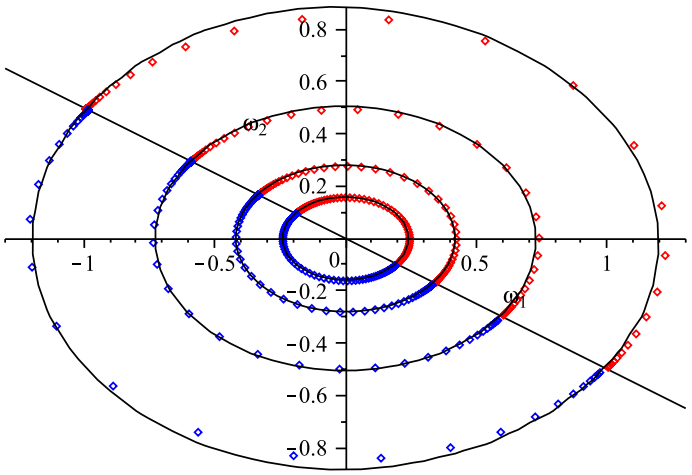
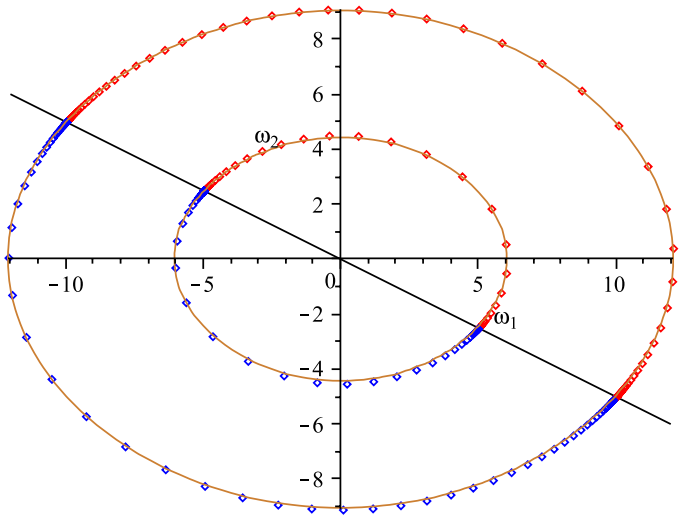


# Asymptotics of conserved quantity (if it exists!)



# Experiments on explicit integrability for the case $I_{11} = I_{22}$

Conserved energy

$$E = \frac{1}{2}\left(M_1^2 + M_2^2\right)$$

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$$\Delta_k = t_{k+1} - t_k, \qquad \text{Constant?}$$

# Experiments on explicit integrability for the case $I_{11} = I_{22}$

$$\Delta_1 = 0.1611957340$$
$$\Delta_2 = 0.1610722014$$
$$\Delta_3 = 0.1609643223$$
$$\Delta_4 = 0.1608883641$$
$$\Delta_5 = 0.1608576431$$
$$\vdots$$
$$\Delta_{80} = 0.16130742$$

Thanks!